week later he hit number 44 to become the franchise leader. By the time he retired after the 1916 season, he had 101 career home runs, including a Pittsburgh record of 82.

Another future Hall of Famer, Paul Waner, succeeded Wagner. Waner closed out 1931 one homer behind Wagner. His first 1935 home run came on May 11 to tie Wagner before passing him two days later with number 83. Waner ended the Pittsburgh portion of his career in 1940 with 109 home runs. While it took Waner 15 years to accumulate his total, his record was broken part way into his successor's third season.

Ralph Kiner, another future Hall of Famer, was the most dominant power hitter in Pittsburgh history to this point. In his rookie season of 1946, he hit 23 to tie the single-season team record. His remaining six full seasons with Pittsburgh produced the six highest season home run totals in team history. In late 1948, on August 26, he hit career home number 109 to tie Waner. September 2nd he passed Waner. Playing part way into 1953, Kiner finished his Pittsburgh career with 301 home runs, on his way to 369 homers in a career prematurely shortened by back trouble. That gives him more than 46% of all the home runs ever hit by players born in New Mexico.

Kiner's record held until 1973. Willie Stargell, another Hall of Famer, hit a grand slam on July 3, 1973, to tie Kiner. On July 11, Stargell hit number 302 to take the franchise lead which he holds to this day. When he retired in 1982, he had a total of 475 home runs, all with Pittsburgh.

Conclusion

These studies are the third installment of a series I hope to continue. Baseball is unique among sports in the way that statistics play such a central role in the game and the fans' enjoyment thereof. The importance of baseball statistics is evidenced by the existence of the Society for American Baseball Research, a scholarly society dedicated to studying baseball.

References and Acknowledgements

This work is made much easier by Lee Sinins' Complete Baseball Encyclopedia, a wonderful software package, and www.baseball-reference.com. It would have been impossible without the wonderful websites www.retrosheet.org and www.sabr.org which give daily results and information for most major league games since the beginning of major league baseball.

Bernoulli Numbers and Their Applications

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Abstract. Bernoulli numbers, named after a Swiss mathematician, Jakob Bernoulli (1654-1705), crop up in various branches of mathematics such as number theory, topology, combinatorics, and analysis. In this article, we will give a recursive definition of Bernoulli numbers and look at some elementary applications of the numbers.

1 Introduction

There are several special numbers in Mathematics. The reader might have come across amicable numbers, Fibonacci numbers, harmonic numbers, Euler numbers, Lucas numbers, or Genocchi numbers. In this article, we are going to briefly look at special numbers called Bernoulli numbers and discuss some of its applications.

2 Recursive Definition of Bernoulli Numbers

Consider the following function of a complex variable z: $f(z) = \begin{cases} \frac{z}{e^z-1}, \text{ for } z \neq 0 \\ 1, \text{ for } z = 0 \end{cases}$

Solving the equation $e^z - 1 = 0$ for z, we get $z = 2\pi i k$, where k is any integer. Since $\lim_{z\to 0} f(z) = 1$, we conclude that $z = 0$ is a removable singularity of f. Hence, the function f has a Maclaurin series that is valid for $|z| < 2\pi$. We write

$$f(z) = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \tag{1}$$

where B_k are constants to be determined.

We recall that

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots$$

is valid for $|z| < \infty$. Therefore

$$e^z - 1 = z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots.$$

The definition of the function f implies $(e^z - 1)f(z) = z$ so use (1) to obtain

$$z = \left(z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots\right)\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k,$$

which, in expanded form, becomes

$$z = \left(z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + \cdots\right)\left(B_0 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \frac{B_4}{4!}z^2 + \frac{B_5}{5!}z^5 + \cdots\right).$$

Equating the coefficients of the powers of z^n on both sides of the above equation, we obtain the following for $n = 1, 2, 3, 4, 5, \ldots, k$:

$$\begin{cases} B_0 = 1 \\ \frac{1}{2!}B_0 + B_1 = 0 \\ \frac{1}{3!}B_0 + \frac{1}{2!}B_1 + \frac{1}{2!}B_2 = 0 \\ \frac{1}{4!}B_0 + \frac{1}{3!}B_1 + \frac{1}{2!\,3!}B_2 + \frac{B_3}{3!\,2!} + \frac{B_4}{4!} = 0 \\ \vdots \\ \frac{B_0}{0!\,(k+1)!} + \frac{B_1}{1!\,(k!)} + \cdots + \frac{B_{k-1}}{(k-1)!\,2!} + \frac{B_k}{k!} = 0 \end{cases} \tag{2}$$

From this, we readily obtain $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0,$ etc.

We claim that $B_{2k+1} = 0$ for all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Here is how we prove our claim. Since $B_0 = 1, B_1 = -\frac{1}{2}$, and

$$f(z) = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, |z| < 2\pi,$$

we may write

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} z^k \, , |z| < 2\pi.$$

Since

$$\begin{aligned} 1 + \sum_{k=2}^{\infty} \frac{B_k}{k!} z^k &= \frac{z}{e^z - 1} + \frac{z}{2} \\ &= \frac{z}{2}\left(\frac{2}{e^z - 1} + 1\right) \\ &= \frac{z}{2}\left(\frac{2}{e^z - 1} + \frac{e^z - 1}{e^z - 1}\right) \\ &= \frac{z}{2}\left(\frac{2 + e^z - 1}{e^z - 1}\right) \\ &= \frac{z}{2}\left(\frac{e^z + 1}{e^z - 1}\right) \\ &= \frac{z}{2} \coth\left(\frac{z}{2}\right), \end{aligned}$$

which is an even function of z, we conclude that $B_{2k+1} = 0$, for all $k \in \mathbb{N}$. This completes the proof of our claim. We observe that in the process of proving our claim, we have the following serendipity:

(3) $$\frac{z}{2} \coth\left(\frac{z}{2}\right) = 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k} \text{, which is valid for } |z| < 2\pi$$

The numbers $B_0, B_1, B_2, B_4, B_6, \ldots$, defined recursively in (2), are called **Bernoulli numbers**. Here is a table of the first 36 nonzero Bernoulli numbers (Hairer and Wanner [7], p. 161):

k	B_k
0	1
1	$-1/2$
2	$1/6$
4	$-1/30$
6	$1/42$
8	$-1/30$
10	$5/66$
12	$-691/2730$
14	$7/6$
16	$-3617/510$
18	$43867/798$
20	$-174611/330$
22	$854513/138$
24	$-236364091/2730$
26	$8553103/6$
28	$-23749461029/870$
30	$8615841276005/14322$
32	$-7709321041217/510$
34	$2577687858367/6$
36	$-26315271553053477373/1919190$

We have seen that if

$$f(z)=\begin{cases}\dfrac{z}{e^z-1}, \text{ for } z\neq 0\\ 1, \text{ for } z=0\end{cases},$$

then

$$f(z)=\frac{z}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}z^{2k}, |z|<2\pi$$

where $B_{2k}, k=0,1,2,3,\ldots$ are Bernoulli numbers.

3 Historical Computations of Bernoulli Numbers ([16])

According to Wikipedia [16], Jakob Bernoulli calculated B_0 through B_{10}. Lady Ada Lovelace (1815–1852) wrote a first computer program to calculate Bernoulli numbers for the Charles Babbage's analytical machine. Euler (1654–1705) calculated up to B_{30}. In 1840, M. Ohm calculated up to B_{62}. In 1877, J.C. Adams calculated up to B_{124} by using a certain theorem. In 1967, D.E. Knuth and T.J. Buckholtz calculated B_{360}. In 1996, S. Ploufe and G.J. Fee calculated $B_{200,000}$. On July 10, 2002, S. Ploufe calculated $B_{750,000}$. In December, 2002, B.C. Kellner calculated $B_{1,000,000}$. In April, 2008, O. Pavlyk calculated $B_{10,000,000}$. In October, 2008, D. Harvey calculated $B_{100,000,000}$.

4 Some Elementary Applications of Bernoulli Numbers in Mathematics

4.1 Bernoulli Summation Formula

In this section, we will state the Bernoulli Summation Formula and then use it to find the exact sums of some important finite series.

Theorem 1 (*Bernoulli Summation Formula*) Fix $m \in \mathbb{N}$ and for $n \in \mathbb{N}$, let

$$S_m(n) = 1 + 2^m + 3^m + \cdots + n^m.$$

Then

$$S_m(n) = \frac{1}{m+1}n^{m+1} + \frac{1}{2}n^m + \frac{mB_2}{2}n^{m-1} + \frac{m(m-1)(m-2)B_4}{2\cdot 3\cdot 4}n^{m-3} + \frac{m(m-1)(m-2)(m-3)(m-4)B_4}{2\cdot 3\cdot 4\cdot 5\cdot 6}n^{m-5} + \cdots,$$

where the series terminates at the last positive power of n.

Proof. See Katz [8], p. 599. ■

Bernoulli used this formula to find

$$S_{10}(1000) = 91409924241424243424241924242500 \text{ (32 digits!)}.$$

According to Dunham [3], p. 12), Bernoulli claimed that he found this 32-digit number "in less than half a quarter of an hour."

Let us use the Bernoulli Summation Formula to find the exact sums of some important finite series that are very useful in Calculus I.

Example 1. Use Bernoulli Summation Formula to show that

$$1 + 2 + 3 + \cdots n = \frac{1}{2}n(n+1). \tag{4}$$

Solution. Setting $m = 1$ in the Bernoulli Summation Formula, we have

$$S_1(n) = \frac{1}{(1+1)}n^{1+1} + \frac{1}{2}n^1 = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1).$$

Therefore

$$1 + 2 + 3 + \cdots n = \frac{1}{2}n(n+1),$$

as we wished to show. Of course, the easiest way to prove (4) is to use a technique used by Carl Friedrich Gauss (1777–1855) to find the sum of the first 100 natural numbers when he was only 7-years old, and he managed to find the correct answer in less than a minute.

Example 2. Use the Bernoulli Summation Formula to show that

$$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Solution. Setting $m = 2$ in the Bernoulli Summation Formula, we have

$$\begin{aligned} S_2(n) &= \frac{1}{2+1}n^{2+1} + \frac{1}{2}n^2 + \frac{2B_2}{2}n^{2-1} \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + B_{2n} \end{aligned}$$

$$= \frac{1}{6}n(n+1)(2n+1).$$

Therefore

$$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{1}{6}n(n+1)(2n+1),$$

as we wished to show.

Similarly, it can be shown that

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

and

$$1^4 + 2^4 + 3^4 + \cdots n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1).$$

4.2 Maclaurin Series of Certain Functions in Terms of Bernoulli Numbers

In this section, we find the Maclaurin series of certain functions in terms of Bernoulli numbers.

Example 1. Show that

$$z\cot(z) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}z^{2k}, |z| < \pi.$$

Solution. From (3), we know that

(5) $$\frac{z}{2}\coth\left(\frac{z}{2}\right) = 1 + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}z^{2k}, \text{ for } |z| < 2\pi.$$

Replacing z with $2iz$ in (5) and using the identity

$$\coth(iz) = -i\cot z,$$

we obtain

(6) $$z\cot(z) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}z^{2k}, \text{ valid for } |z| < \pi.$$

This is precisely what we wished to show.

Example 2. Show that

$$\tan(z) = \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(1-2^{2k})B_{2k}}{(2k)!}z^{2k-1}, |z| < \frac{\pi}{2}.$$

Solution. It is left as an exercise for the reader to use (6) and the trigonometric identity

$$\tan z = \cot z - 2\cot 2z$$

to obtain

$$\tan(z) = \sum_{k=1}^{\infty}(-1)^k \frac{2^{2k}(1-2^{2k})B_{2k}}{(2k)!}z^{2k-1}, |z| < \frac{\pi}{2}.$$

4.3 Relationship with the Riemann Zeta Function $\zeta(z) = \sum_{k=1}^{\infty} 1/k^2$, $\mathrm{Re}(z) > 1$

In this section, we state a result that gives a relationship between Bernoulli numbers and the Riemann zeta function.

Theorem 2. If n is a natural number, then

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}.$$

Proof. For an elegant proof, see, for example, Simmons [13], p. 302. ∎

Example

Value of n	Sum of the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$
1	$\pi^2/6$
2	$\pi^4/90$
3	$\pi^6/945$
4	$\pi^8/9450$
5	$\pi^{10}/93555$
6	$\pi^{12}/638512875$

Remark. We need another Euler to evaluate

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}},$$

where $n \in \mathbb{N}$.

5 Conclusion

Although we have only looked at three applications of the Bernoulli numbers in this article, there are many applications of Bernoulli numbers in various branches of mathematics such as number theory, topology, combinatorics, and analysis.

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Biographical Sketch

Lloyd Moyo received his B.Ed. (Science) in 1992 from the University of Malawi in Southern Africa. He received his M.Sc. in Mathematics from the University of Sussex, U.K. in 1996 and his Ph.D. in Mathematics from New Mexico State University in 2006. He joined Henderson State University in fall 2012. He is a member of the American Mathematical Society, the Mathematical Association of America, Arkansas Academy of Science, and the International Mathematical Union.

Library Funding at Colleges and Universities in the United States

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In an era of flat or reduced budgets, libraries at institutions of higher education across the country struggle to continue to offer a quality level of service to their patrons while the prices of materials continue to increase. To meet the needs of students, faculty, researchers, and other stakeholders, libraries are finding creative solutions to access the materials that are critical to colleges and universities. This paper examines the ways that libraries are trying to solve this funding problem and how they respond to budget cuts.

Library Funding at Colleges and Universities in the United States

With increasing prices for electronic materials required by faculty and students, college and university libraries around the world are struggling to meet these growing demands. Using a critical eye on current holdings and exploring ways to save money when purchasing new materials, these libraries can continue to offer these necessary resources.

Literature Review

The study of budget cuts on libraries at an institution of higher education is limited. Often it is included as an aside in stories that focus on other cuts at institutions. Other publications focus on the impact of cuts at public libraries rather than academic institutions. One such article is Kelly (2011), where the author laments the status of library budgets, writing "examining *Library Journal*'s annual budget survey is like scanning a battlefield: there are bodies everywhere, the smoke and dust are blinding."

Some of the best resources on the status of library funding come from the American Library Association's *State of America's Libraries* (2011). Other publications include Lyall and Sell's (2006) impact of cuts to public higher education and in particular the impact it has had on libraries.

With such a wide variety of topics that are directly related to academic library finance, it is difficult for one publication to cover all aspects of the subject. Thus a single reference has not yet been published that does that.