professional journals and socio-cultural venues including *The Old Time Chronicle, The Southern Standard, The Journal of Poetry Therapy* and *Tales from the South.*

Linda lives in the farmhouse; which is the setting for many of her stories with Buford and Babe, her silver-point tabby and black Labrador retriever, respectively.

Evaluation of the Dirichlet Integral by a Fourier Transform Method

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Abstract. The improper integral

$$\int_0^\infty \frac{\sin x}{x}dx$$

is known as the Dirichlet integral, named after Johann Peter Gustav Lejeune Dirichlet (1805-1859), a German mathematician. In this article, we will show, by using a Fourier transform method indirectly, that

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Introduction

The function $h(x) = \dfrac{\sin x}{x}$ crops up in undergraduate and graduate mathematics courses. It is called a sinc function and is written as

$$\text{sinc } x = \frac{\sin x}{x}.$$

The sinc function has many interesting properties. In this article, we evaluate the Dirichlet integral, $\int_0^\infty \frac{\sin x}{x}dx$, associated with the sinc function, by using a Fourier transform method. We will conclude by sketching a proof that

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}.$$

Basic Definitions

Definition 1 (R. K. Nagle, E.B. Saff & A.D. Snider [3], p. 355) A function *f* is said to be piecewise continuous on the interval $[a,b]$ if *f* is continuous at every point in $[a,b]$ except possibly for a finite number of points at which *f* has a jump discontinuity. A function *f* is said to be piecewise continuous on the interval $[0,\infty)$ if *f* is piecewise continuous on the interval $[0,\alpha]$ for all $\alpha > 0$.

Definition 2 A function *f* is said to be absolutely integrable on the interval $(-\infty,\infty)$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Definition 3 Suppose *f* is a piecewise continuous and absolutely integrable function on $(-\infty,\infty)$. The Fourier transform of *f*, denoted by $\mathcal{F}\{f\}(\omega)$, or $\hat{f}(\omega)$, is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)\,dx.$$

The inverse Fourier transform of $\hat{f}(\omega)$ denoted by $f(x)$ defined by

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i\omega x} \hat{f}(\omega) d\omega. \tag{1}$$

If *f* has a jump discontinuity at the point *x*, then we replace the left-hand side of (1) by $\frac{f(x+)+f(x-)}{2}$ where $f(x+)$ denotes the limit of $f(t)$ as *t* approaches *x* from the right side and $f(x-)$ denotes the limit of $f(t)$ as *t* approaches *x* from the left side. Thus, we have a Fourier transform pair

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)dx \leftrightarrow f(x) = \int_{-\infty}^{\infty} e^{2\pi i\omega x} \hat{f}(\omega)\, d\omega.$$

Universality Question for Fourier Transform Definition

Definition 3 of Fourier Transform is not universal. If we let $\alpha = \frac{1}{\sqrt{2\pi}}$, other commonly used Fourier transform pairs are:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{2\pi i\omega x} f(x)dx \leftrightarrow f(x) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} \hat{f}(\omega)d\omega$$

$$\hat{f}(\omega) = \alpha\int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \leftrightarrow f(x) = \alpha\int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)d\omega$$

$$\hat{f}(\omega) = \alpha\int_{-\infty}^{\infty} e^{i\omega x} f(x)dx \leftrightarrow f(x) = \alpha\int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega)d\omega$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx \leftrightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)d\omega$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)dx \leftrightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega)d\omega$$

Example of Fourier Transform of a Function

Example 1. Let $b > 0$. Consider the following unit rectangular pulse function *f*, where

$$f(x) = \begin{cases} 1, & |x| \le b, \\ 0, & |x| > b. \end{cases}$$

Find the Fourier transform $\hat{f}(\omega)$ of *f*.

Solution

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} f(x)dx = \int_{-b}^{b} e^{-2\pi i\omega x} dx$$

$$= \frac{-1}{2\pi i\omega}\left(e^{-2\pi i\omega b} - e^{2\pi i\omega b}\right) = \frac{1}{\pi\omega}\left(\frac{e^{2\pi i\omega b} - e^{-2\pi i\omega b}}{2i}\right) = \frac{1}{\pi\omega}\sin(2\pi b\omega).$$

Evaluation of the Dirichlet Integral by a Fourier Transform Method

The improper integral

$$\int_0^\infty \frac{\sin x}{x}dx$$

is known as the Dirichlet integral, named after Johann Peter Gustav Lejeune Dirichlet (1805-1859), a German mathematician. In this section, we will show, by using a Fourier Transform Method indirectly, that

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Here is how we show this. From **Example 1**, we know that if $b > 0$ and $f(x) = \begin{cases} 1, & |x| \le b \\ 0, & |x| > b \end{cases}$, so the Fourier transform of f is given by

$$\hat{f}(\omega) = \frac{1}{\pi\omega}\sin(2\pi b\omega).$$

Using the inverse Fourier transform, for $|x| < b$, we have

$$f(x) = \boldsymbol{\mathcal{F}^{-1}}\left\{\frac{1}{\pi\omega}\sin(2ib\omega)\right\}(x) = \int_{-\infty}^{\infty} \frac{1}{\pi\omega}\sin(2\pi b\omega)e^{2\pi i\omega x}\, d\omega$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi b\omega)}{\omega}e^{2\pi i\omega x}d\omega.$$

Up to this point, we have shown that if $|x| < b$, then

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi b\omega)}{\omega}e^{2\pi i\omega x}d\omega. \tag{2}$$

Setting $x = 0$ in (2), we have

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi b\omega)}{\omega}d\omega.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi b\omega)}{\omega}d\omega = \pi. \tag{3}$$

Since the integrand $\dfrac{\sin(2\pi b\omega)}{\omega}$ in the left-hand side of (3) is an even function of ω, equation (3) becomes

$$2\int_0^\infty \frac{\sin(2\pi b\omega)}{\omega}d\omega = \pi,$$

from which we have

$$\int_0^\infty \frac{\sin(2\pi b\omega)}{\omega}d\omega = \frac{\pi}{2}. \tag{4}$$

By setting $b = \frac{1}{2\pi}$ in equation (4) we obtain

$$\int_0^\infty \frac{\sin(\omega)}{\omega} d\omega = \frac{\pi}{2}.$$

Since ω is a dummy variable, we conclude that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

This completes the evaluation of the Dirichlet integral by using a Fourier transform method.

In Case You Need a Take-home Exam

1. Let $f(x) = e^{-b|x|}$, where $b > 0$. Show that the Fourier transform of *f* is given by

$$\hat{f}(\omega) = \frac{2b}{b^2 + 4\pi^2\omega^2}.$$

Hence, deduce that

$$\int_0^\infty \frac{\cos(\omega x)}{b^2 + x^2} dx = \frac{\pi}{2b} e^{-b|\omega|}.$$

An Elegant Associated Integral

We have the following surprising result:

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

To prove this, we take the function *f*, where

$$f(x) = \begin{cases} 1, & |x| \le \frac{1}{2\pi}, \\ 0, & |x| > \frac{1}{2\pi}. \end{cases}$$

From **Example 1**, we know that its Fourier transform is given by

$$\hat{f}(\omega) = \frac{1}{\pi\omega} \sin \omega.$$

Then the result follows by using the following mathematical bazooka, namely:

Theorem 4 (Plancherel's Identity): Suppose that *f* is a piecewise continuous and absolutely integrable function on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. Then

$$\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega < \infty \text{ and } \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 d\omega.$$

Proof. See, for example, A. Pinkus and S. Zafrany [4], p. 114.

Biographical Sketch

Lloyd Moyo received his B.Ed (Science) in 1992 from the University of Malawi in southeastern Africa. He received his M.Sc. in Mathematics from the University of Sussex, U.K. in 1996 and his Ph.D. in Mathematics from New Mexico State University in 2006. He joined Henderson State University in fall 2012. He is a member of the American Mathematical

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