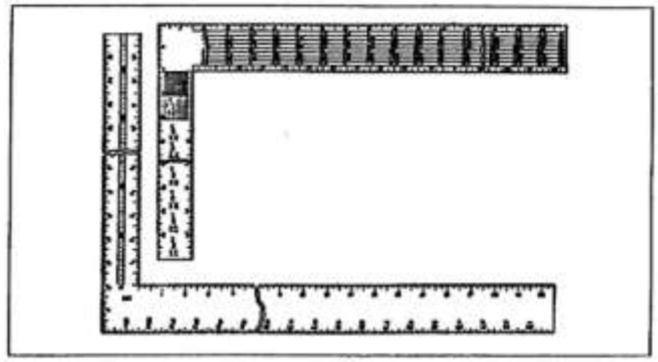
Mathematics of a Carpenter's Square

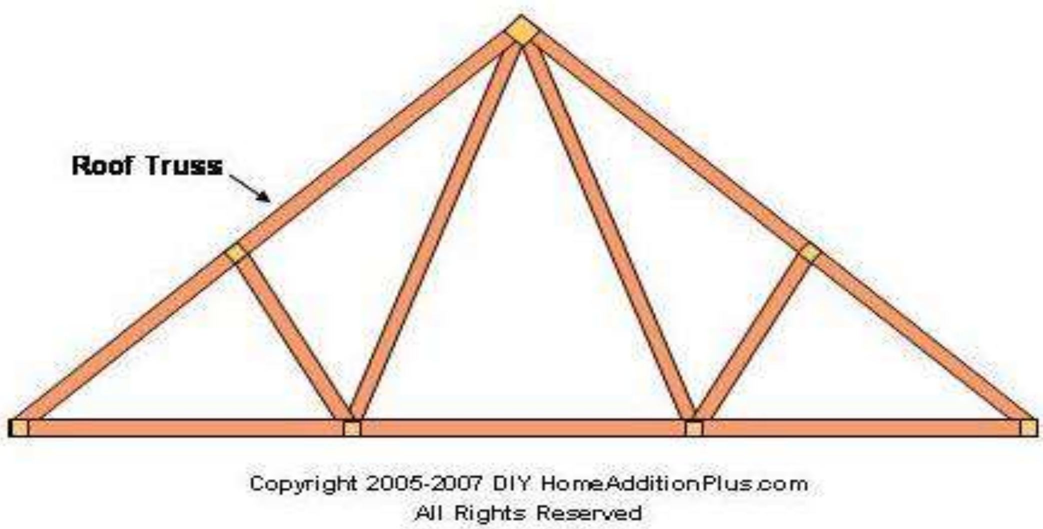
Michael Lloyd, Ph.D.
Professor of Mathematics

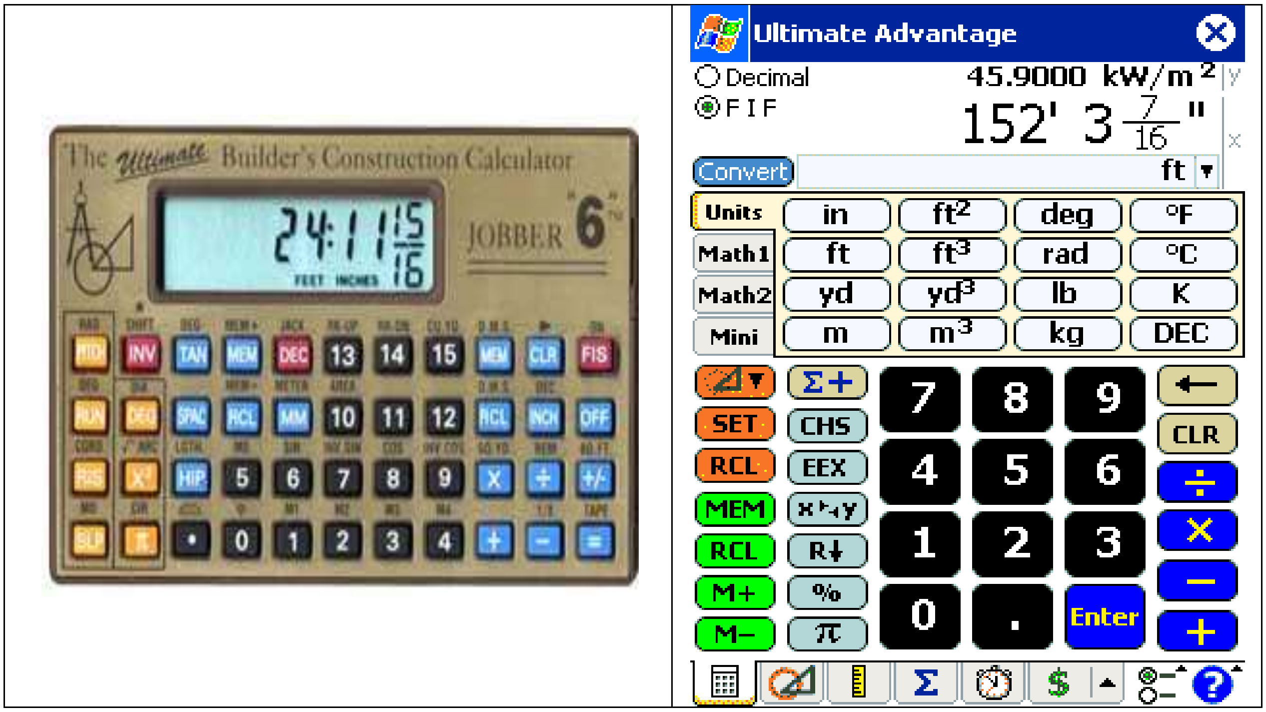
Abstract
The mathematics behind extracting square roots, the octagon scale, polygon cuts, trisecting an angle and other techniques using a carpenter's square will be discussed.

Introduction
The carpenter's square was invented centuries ago, and is also called a builder's, flat, framing, rafter, and a steel square. It was patented in 1819 by Silas Hawes, a blacksmith from South Shaftsbury, Vermont. The standard square has a 24 x 2 inch blade with a 16 x 1.5 inch tongue.



Using the tables and scales that appear on the blade and the tongue is a vanishing art because of computer software, construction calculators, and the availability of inexpensive prefabricated trusses.





Although practically useful, the Essex, rafter, and brace tables are not especially mathematically interesting, so they will not be discussed in this paper.

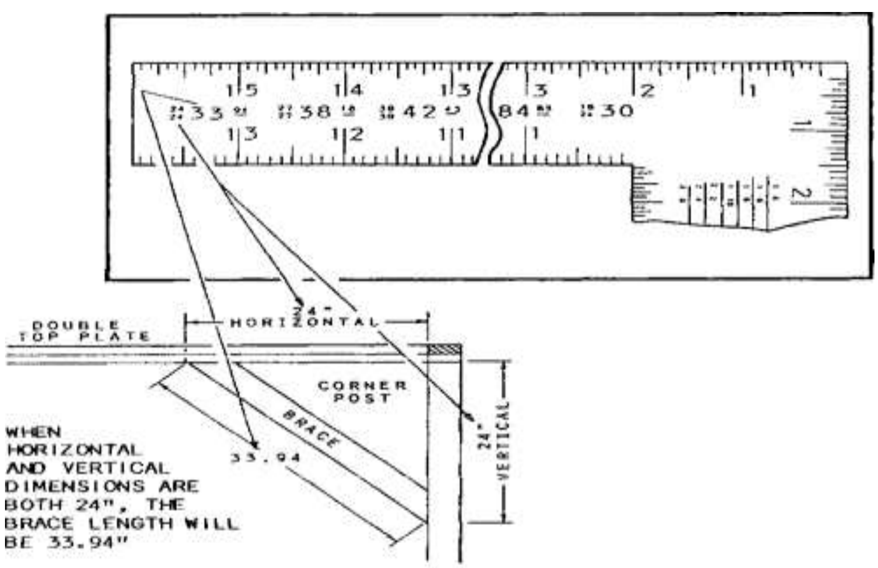
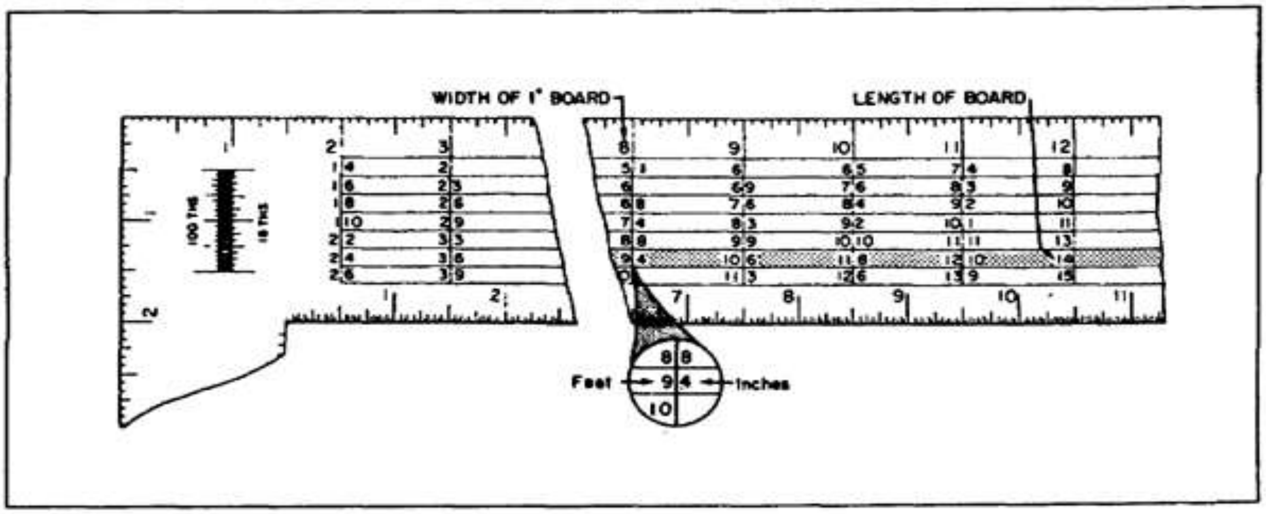
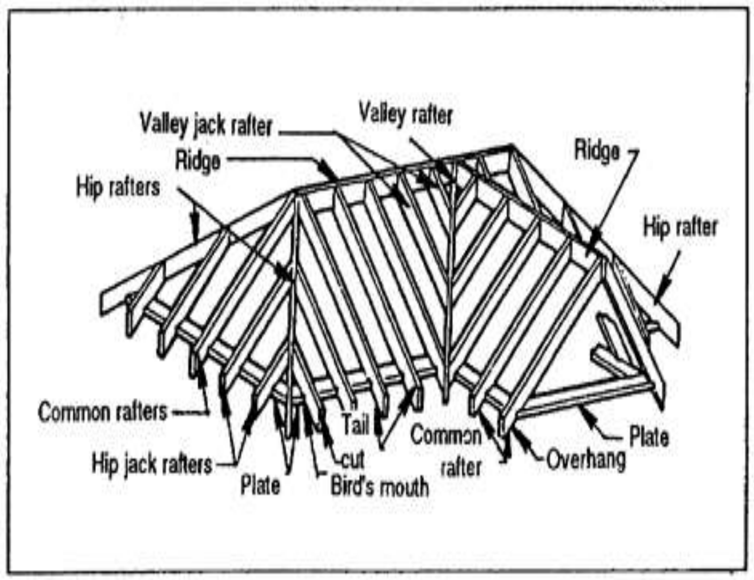


Figure 2-11.-Brace table.





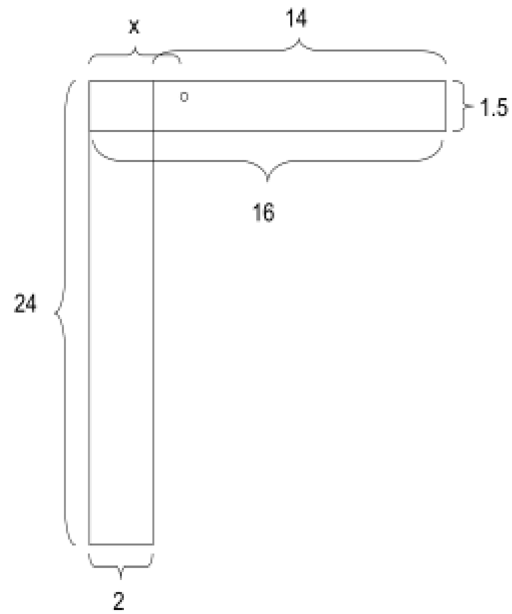
Balanced Peg Hole

Some squares have a small hole drilled into them so that the square may be hung on a nail. Where should the hole be drilled so the blade will be vertical when the square is hung? We will derive the optimum location of the hole, *x*, as measured from the corner along the edge of the blade.

Assume that the hole removes negligible material. The center of the blade is 1" from the left, and the center of the tongue is (2+16)/2 = 9" from the left. Thus, the centroid, and hence the center of mass, is

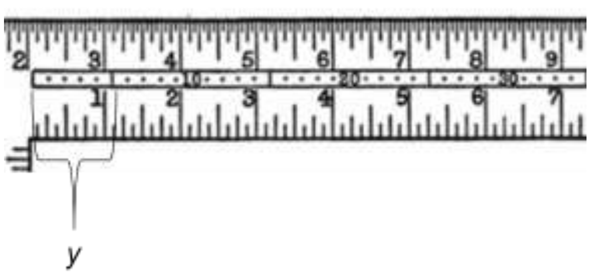
$$x = \frac{1'' \times 2 \times 24 + 9'' \times 1.5 \times 14}{2 \times 24 + 1.5 \times 14} = 3\frac{10}{23}'' \approx 3.43''$$

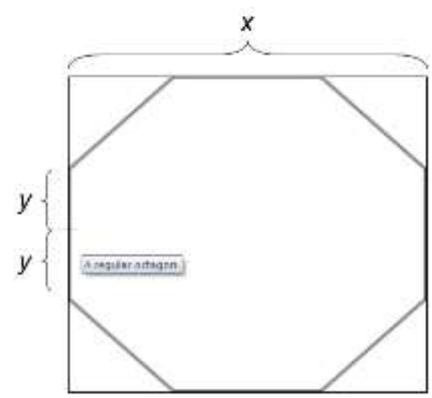
I thought that I had made an error since my square measured $x = 3.6$ inches. However, when I hung my square, the blade was not exactly vertical.



Octagon Scale

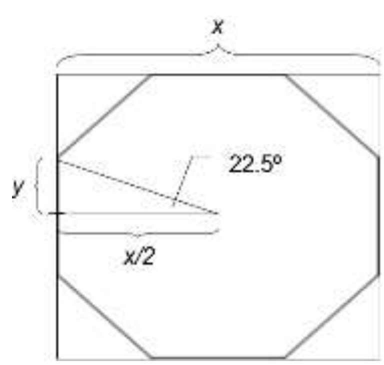
This scale is useful for shaping octagon posts. Suppose a square has side length *x* in inches. Find this on the scale and measure *y* as shown on the first figure with a compass or tape ruler. The distance from the center of an edge to nearest vertex of the octagon is *y* as shown in the second figure.





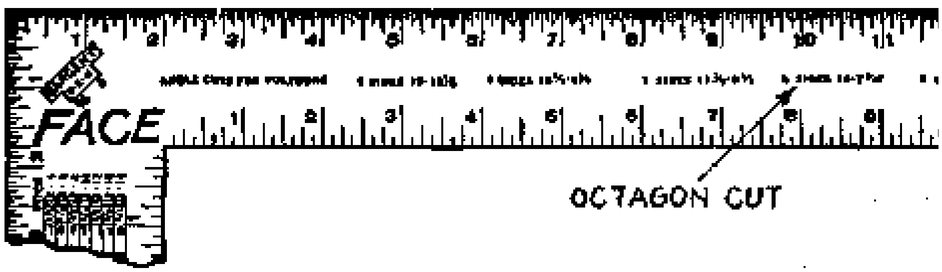
For example, suppose *x* is 5 inches, then find this on the scale, and measure out from the center of each side distance *y*, about 1.125 inches.

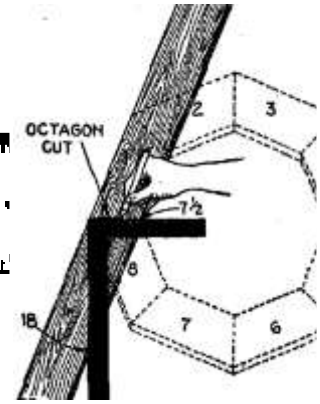
The construction of this scale follows from relating the legs of the right triangle shown here. To construct large octagons, it is more practical to simply multiply *x* by 0.207 to obtain y.



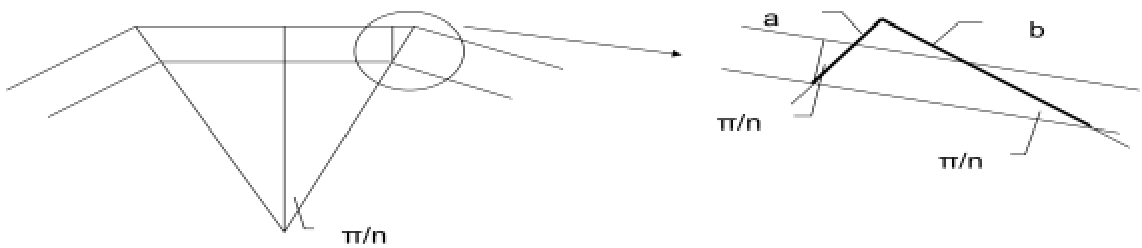
$$y = \frac{x}{2}\tan 22.5^\circ = \frac{\sqrt{2}-1}{2}x \approx 0.207x$$

Polygon Cuts
These are alternating cuts made on a board so as to construct a regular polygon as shown in the accompanying figure.





Suppose we want to create an n-gon where n is at least 5. The polygon table is constructed by finding rational numbers a and b with denominators equaling 8 so that $\tan\frac{\pi}{n} \approx \frac{a}{b}$:



The top three rows of the following table were obtained from the barely legible polygon table in Audel's. The last row measures how well a/b approximates the theoretical tangent value.

Sides	5	6	7	8	9	10
Tongue a	13 1/8	9 5/8	8 5/8	7 1/2	6 1/8	5 7/8
Blade b	18	16 5/8	17 7/8	18	16 3/4	18
a/b	0.729	0.579	0.483	0.417	0.366	0.326
tan π/n	0.727	0.577	0.482	0.414	0.364	0.325
Relative Error	$3x10^{-3}$	$3x10^{-3}$	$2x10^{-3}$	$7x10^{-3}$	$5x10^{-3}$	$3x10^{-3}$

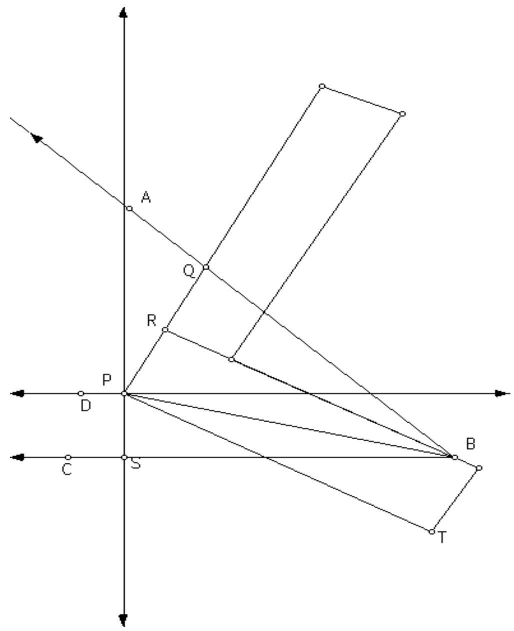
Using a TI-89 program that searches for the best solution by an exhaustive search and requiring that the blade length be between 16 and 18 inches gave the following table. I do not know why these values are better than the ones given in Audel's.

Sides	5	6	7	8	9	10
Tongue a	11 5/8	10 1/4	8 1/8	7 1/4	6 3/8	5 1/4
Blade b	16	17 3/4	16 7/8	17 1/2	17 1/2	16 1/8
a/b	0.7266	0.5775	0.4815	0.4143	0.3643	0.3256
tan π/n	0.7265	0.5774	0.4816	0.4142	0.3640	0.3249
Relative Error	$3x10^{-5}$	$2x10^{-4}$	$2x10^{-4}$	$2x10^{-4}$	$9x10^{-4}$	$2x10^{-3}$

Trisecting an Angle

To trisect the Angle ABC,

1) Draw the line through DP 2" higher and parallel to CB
2) Position the carpenter's square so PQ = 4". Then R will be at the 2" mark.
3) Then Angle QBR = Angle RBP = Angle PBC

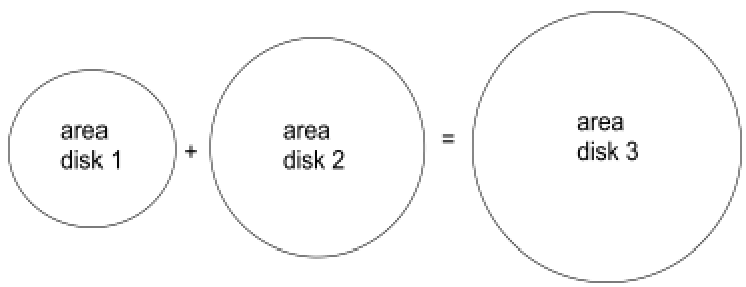


Here is a derivation of this method:

1) PS = 2" = RP and angle PSB = 90º = Angle PRB. Since they share the same hypotenuse, namely PB, the triangles PSB and PRB are congruent.
2) Similarly, the right triangles PRB and QRB are congruent.
3) Thus, Angle SBP = Angle PBR = Angle RBQ. This is the trisection of Angle QBS = Angle ABC.

Summing Disks

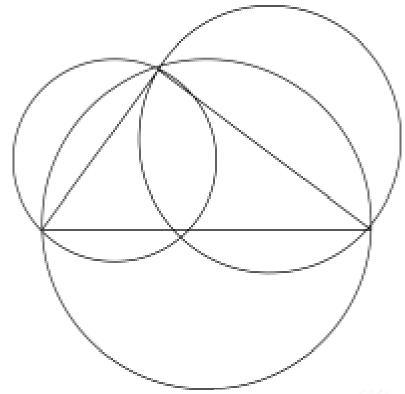
The square can be used to construct the disk whose area is the sum of two given disks.



For a proof, let diameters of the disks be d_1, d_2, d_3, respectively. Then the following computation shows that the diameters satisfy the Pythagorean Theorem:

$$\text{area } 1 + \text{area } 2 = \text{area } 3$$
$$\frac{\pi}{4}d_1^2 + \frac{\pi}{4}d_2^2 = \frac{\pi}{4}d_3^2$$
$$d_1^2 + d_2^2 = d_3^2$$

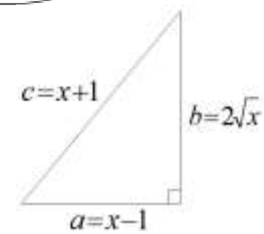
In practice, the diameters can be measured using the tongue and blade, and then nails can be driven at the ends of the diameter of the sum disk. Rotate the square and trace out the disk by holding your pencil at the right-angle vertex.



Square Roots

To find the square root of x, create a right triangle with one leg equaling x-1, and the hypotenuse equaling x+1. Measure the other leg, and divide by 2. The following computation which starts with the Pythagorean Theorem explains why this method works:

$$b^2 = c^2 - a^2 \\ = (x+1)^2 - (x-1)^2 \\ = 4x$$



To estimate the accuracy of this method, take the differential of the above Pythagorean formula to obtain

$$db = \frac{cdc - ada}{b} = \frac{(x+1)dc - (x-1)da}{2\sqrt{x}}$$

If the precision of each measurement is 1/32 ", then the error from drawing the triangle is

$$|db| \le \frac{x+1}{2\sqrt{x}}|dc| + \frac{x-1}{2\sqrt{x}}|da| \le \sqrt{x}\left(\frac{1''}{32}\right)$$

Side b is also measured, so the maximum relative error for computing a square root is

$$\left|\frac{d\sqrt{x}}{\sqrt{x}}\right| \le \left(\frac{\sqrt{x}}{2} + 1\right)\left(\frac{1}{32}\right) \div \sqrt{x} = \left(\frac{1}{2} + \frac{1}{\sqrt{x}}\right)\left(\frac{1}{32}\right) \approx 2 - 3\%$$

It is actually possible to solve a more general problem: Extract the roots of the quadratic equation $ax^2 + bx + c = 0$ with positive coefficients. The java applet at http://www.concentric.net/~pvb/GEOM/complexquad.html shows geometrically how to extract the real or non-real roots.

References

- Audels Carpenters and Builders Guide, ©1964 Theodore Audel & Co.
- http://64.78.42.182/sweethaven/BldgConst/Carpentry01
- http://www.aquapoolsandspas.com/gazebos/latticeoctagon.html
- http://www.bobvila.com/HowTo_Library/The_Carpenter_s_Square-Hand_Tools-A2046.html
- http://www.concentric.net/~pvb/GEOM/complexquad.html
- http://harvardforest.fas.harvard.edu/museum/Sanderson/sanderson12.html
- http://www.homeadditionplus.com/framing-info/Roof-Trusses-vs-Roof-Rafters.htm
- http://www.jobbercalculator.com/
- http://journeytoforever.org/farm_library/device/devices2.html
- http://mathforum.org/geopow/archive/19970606.geopow.html
- http://www.taunton.com/finehomebuilding/how-to/articles/laying-out-octagons.aspx
- http://www.waybuilder.net/sweethaven/BldgConst/Building01/default.asp?iNum=0902
- My father, Steven Lloyd

Biography

Michael Lloyd received his B.S in Chemical Engineering in 1984 and accepted a position at Henderson State University in 1993 after earning his Ph.D. in Mathematics from Kansas State University. He has presented papers at meetings of the Academy of Economics and Finance, the American Mathematical Society, the Arkansas Conference on Teaching, the Mathematical Association of America, and the Southwest Arkansas Council of Teachers of Mathematics. He has also been an AP statistics consultant since 2001 and a member of the American Statistical Association.