"But I Got the Right Answer"

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Abstract - Students often make mistakes in working problems. Occasionally those mistakes end up producing the correct answer anyway. The purpose of this paper is to look at a few common errors students make and see when those errors can produce correct answers.

Introduction

The purpose of this talk is to take common errors students make and seeing if they are ever actually valid approaches to the problem. Some are only trivially valid. For instance, sometimes students err in applying the distributive property as follows.

$$a\,(x + b) = ax + b$$

Since the correct application would be

$$a\,(x + b) = ax + ab$$

the error producing a correct result would require

$$b = ab \text{ or } b - ab = 0 \text{ or } b\,(1 - a) = 0.$$

Thus, this only works if $b = 0$, in which case we're not really using the distributive property, or $a = 1$, which means there is not a whole lot to the problem.

In this talk I will look at some errors to see if they are only correct under trivial circumstances or if there are actually some non-trivial times where they work. I will not consider cases where students make multiple errors accidentally resulting in correct answers. I will only consider "methods" that some students regularly use.

I will make no effort to look at problems where students make two or more common errors. The possibilities there are endless.

Fractions

Fractions are one of the greatest sources for common errors. I believe there are a couple of reasons for this.

First, they are encountered fairly early in a student's education. Elementary school is where students first start seeing fractions. Bad habits can develop there and, if not caught quickly, those habits become entrenched.

Secondly, many people are convinced that fractions are a hard entity with which to work. My belief is that this is because many people do not learn what words in mathematics really mean. Words like "factor" and "term," though similar, have many differences that can create problems in later learning. Fractions are one place where this confusion rears its head.

Lastly, fractions cause people problems because they often do not learn concepts that will get them beyond their intuitive understanding of fractions. Every little child understands getting half of a candy bar or a third of a piece of cake. This intuitive understanding is great but it is also deceptive. I think this intuition lulls people into a sense of complacency that causes them to miss the more complicated aspects of fractions.

Errors

I want to start off with an error that I have never actually witnessed anyone making. I include it, however, for two reasons. First, it is one of the main inspirations for this book, that such an obviously wrong method can still give a correct answer sometimes. Secondly, I once showed it to a colleague who informed me of a mathematics instructor she knows who would give full credit, even for this error, because "the answer is right."

1. $\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$ $\qquad$ $\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}$

This error has the cruel trick of being related to a legitimate approach to such problems. After all, when a fraction has something in the numerator and denominator that thing can be "cancelled." That illustrates why I despise the word "cancel" for use in such situations. I prefer using words that actually describe what is being done to lessen possible confusion.

Now let us see how often this error will actually lead to a correct answer. One could simply program Excel or some other spreadsheet to try all possible fractions and see which ones work. I will use that method some in other examples but in this one I choose to be more mathematical. First we will define the "rules." We are looking for a pair of two digit numbers in which one of the digits of the numerator matches one of the digits of the denominator.
We'll start with the ones digit of the numerator matching the tens digit of the denominator. Then we are looking at a situation where, for integers a, b and c, from 1 to 9, we have the following.

$$\frac{10a+b}{10b+c}=\frac{a}{c}$$

$(10a + b)\ c = (10b + c)\ a$
$10ac + bc = 10ab + ac$
$9ac = 10ab - bc$
$9ac = b\ (10a - c)$

Since 9 divides the left side, it must also divide the right side. We will now consider three cases.

(i) $9 \mid b$

Then $b = 9$.
$ac = 10a - c$
$c = 10a - ac$
$c = (10 - c)\ a$
Since $10 - c \mid c$, we must have $c = 5$, 8 or 9.

If $c = 5$, then $a = 1$. (1)
If $c = 8$, then $a = 4$. (2)
If $c = 9$, then $a = 9$. (3)

Three "solutions" exist so far.

Line (1) gives $\frac{19}{95}$

Line (2) gives $\frac{49}{98}$

Line (3) gives $\frac{99}{99}$

The second and third "solutions" still would require some legitimate reducing to lowest terms but they work.

(ii) $9 \mid (10a - c)$

Then $10a - c$ must be 9, 18, 27, 36, 45, 54, 63, 72, or 81. Each of those are possible with $a = c = 1, 2, 3, 4, 5, 6, 7, 8,$ or 9, respectively. Also, we then have

$9ac = b\,(10a - c)$
$9a^2 = b\,(10a - a)$
$9a^2 = 9ab$

requiring that $a = b = c$. This gives 9 solutions, though only 8 new ones.

$$\frac{11}{11}, \frac{22}{22}, \frac{33}{33}, \frac{44}{44}, \frac{55}{55}, \frac{66}{66}, \frac{77}{77}, \frac{88}{88}$$

(iii) $3 \mid b$ and $3 \mid (10a - c)$

Then $b = 3$ or 6, with 9 being taken care of in case (i).

If $b = 3$, then $3ac = 10a - c$. Then $c = 10a - 3ac = a\,(10 - 3c)$ forcing $(10 - 3c) \mid c$. This is only possible if $c = 3$ giving $a = 3$ and $b = 3$ which is not a new solution.

If $b = 6$, then $3ac = 2\,(10a - c)$. So $3ac = 20a - 2c$. Solving for c yields the following.

$$c = \frac{20a}{3a + 2}$$

Our restrictions on a, b and c force $a = 1$, 2 or 6, giving $c = 4$, 5 or 6 respectively. This produces the following.

$\frac{16}{64}$, $\frac{26}{65}$, and $\frac{66}{66}$. The last one is not new.

Similar methods can then be used in cases where the ones digit of the denominator matches the tens digit of the denominator, the two ones digits match, and the two tens digits match. The first of these settings yields exactly the reciprocals of the ones we have already found. The last two settings yield only fractions which are equal to 1, nine of which we had already found.

So, how many solutions exist? There are 81 two digit numbers that do not have 0 as a digit. Thus there are 81 such fractions equal to 1. And we found eight others giving a total of 89. Then, if a fraction is randomly chosen using our constraints, the probability is then $89/81^2 \approx .0136$ that this "method" will result in a correct reduction to lowest terms.

Interestingly, however, only four of the cases require no further simplification. These are the following.

$$\frac{16}{64}, \frac{26}{65}, \frac{19}{95}, \frac{65}{26}$$

2. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a+c}{b+d}$

The next example is, sadly, one that I see regularly in my intermediate algebra classes.

We will assume that *a* and *b* are relatively prime. Likewise, *c* and *d*. Also, we will assume that *b* and *d* are both positive. Since $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{b\,d}$ for this error to actually return a correct answer gives $\frac{a+c}{b+d} = \frac{ad+bc}{b\,d}$. Now for a little algebra.

$$\frac{a+c}{b+d} = \frac{ad+bc}{b\,d}$$

$$(a+c)\,b\,d = (ad+bc)\,(b+d)$$

$$abd + bcd = abd + ad^2 + b^2c + bcd$$

$$0 = ad^2 + b^2c$$

$$-a = \frac{b^2\,c}{d^2} \text{ and } -c = \frac{d^2\,a}{b^2}$$

From these last two conditions, and the fact that the original fractions were in lowest terms we have that *d* must divide *b* and *b* must divide *d*. Hence *d* = *b*. Then, we must have -*a* = *c*. In other words, the sum of the fractions, correctly and incorrectly, is 0.

3. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a+c}{b\,d}$

Sometimes students remember that adding fractions requires that they do something with the denominators before they add. This is the result. Clearly this "method" will work under three circumstances.

a.) $b = d = 1$
b.) $a = c = 0$
c.) $a = -c$, $b = d$

The question is whether or not there are other cases where it works.

$$\text{Suppose } \frac{a}{b} + \frac{c}{d} = \frac{a+c}{b\,d}.$$

$$\frac{a\,d}{b\,d} + \frac{b\,c}{b\,d} = \frac{a+c}{b\,d}$$

$$\frac{a\,d + b\,c}{b\,d} = \frac{a+c}{b\,d}$$

$$a\,d + b\,c = a + c$$

$$a\,(d-1) = c\,(1-b)$$

$$a = \frac{c\,(1-b)}{d-1} \qquad (*)$$

Using Excel I got the following probabilities of the "method" working by trying all possible combinations of *a*, *b*, *c*, and *d* subject to the given constraints.

$0 < a, b, c, d < 10$	0.0123
$0 < a, b, c, d < 10$, $(a, b) = 1 = (c, d)$, $b > 1$, $d > 1$	0
$0 < b, c < 10$, $1 < d < 10$, a as in (*) an integer	0.5308

The last scenario is the most interesting. It is satisfied by fractions such as -12/9 and 9/7. Below we show the sum, first correctly, second incorrectly.

$$\frac{-12}{9} + \frac{9}{7} = \frac{-84+81}{63} = \frac{-3}{63} = \frac{-1}{21}$$

$$\frac{-12}{9} + \frac{9}{7} = \frac{-12+9}{63} = \frac{-3}{63} = \frac{-1}{21}$$

Of these cases, 72 had a = 0 and 113 others had either *a* and *b* not relatively prime or *c* and *d* not relatively prime. One pair that has both fractions in lowest terms, and works, is -16/9 and 8/5.

$$\frac{-16}{9} + \frac{8}{5} = \frac{-80+72}{45} = \frac{-8}{45}$$

$$\frac{-16}{9} + \frac{8}{5} = \frac{-16+8}{45} = \frac{-8}{45}$$

The probability that the fractions end up in lowest terms and $a \neq 0$ is approximately 0.2454.

As a last test for this "method," I generated 10,000 random pairs of fractions of integers, with the integers ranging from 1 to 50. Because of formula * above, it is clear that unless b = 1, a < 0 is necessary for it to work, I will restrict a to be negative. Only 19 out of the 10,000 pairs worked using this "method" of adding fractions. Six of those fit case a.) from above, where both denominators are 1. Two were case c.). The other 11 were as follows.

i) -4/6, 16/21
ii) -14/7, 42/19
iii) -27/7, 18/5
iv) -15/16, 29/30
v) -45/46, 22/23
vi) -1/12, 2/23
vii) -12/31, 2/6
viii) -6/7, 23/24
ix) -19/39, 1/3
x) -49/50, 10/11
xi) -45/21, 36/17

Pairs iii), iv), v), vi), viii), ix) and x) fit the preference that fractions be in lowest terms.

It is clear that there are non-trivial cases where this "method" works, however, the probability of it working is so small it is not likely an instructor will have very many of them come up in the course of a teaching career.

10. At times it is necessary to break a fraction up into two separate fractions. Not surprisingly, this can cause problems such as the following common error.

$$\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$$

Having a = 0 works trivially so assume it is non-zero. Multiplying both sides by bc (b + c) gives the following.

$$a\,b\,c = a\,(b + c)\,c + a\,(b + c)\,b$$
$$a\,b\,c = a\,b\,c + a\,c^2 + a\,b^2 + a\,b\,c$$
$$0 = a\,c^2 + a\,b^2 + a\,b\,c$$
$$0 = c^2 + b^2 + b\,c$$
$$0 = c^2 + b\,c + b^2$$

Obviously, either b or c must be negative. If $|b| \geq |c|$, then $b^2 \geq |b\,c|$ so the final equation is an impossibility, leaving only the trivial solution, a = 0.

ORDER OF OPERATIONS

One topic causes students great trouble even though we give them cute ways of remembering the rules. We teach them to "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" to help them remember that things in **P**arentheses are done first, then **E**xponents, then **M**ultiplication and **D**ivision, then **A**ddition and **S**ubtraction.

1.) a + b c + d = (a + b)(c + d)

Here students often decide to do the additions first, then do the multiplications. For this to work we would need the following.

$$a + bc + d = ac + ad + bc + bd$$
$$a + d = ac + ad + bd$$
$$a - ac - ad = bd - d$$
$$a (1 - c - d) = d (b - 1)$$

We end up with several cases.

(i) If a = 0, then it will work if either d = 0 or b = 1, but this is pretty much a trivial case.

(ii) If 1 - c - d = 0 (or c + d = 1), then it will work if either d = 0 or b = 1. If d = 0, then c = 1 and we have a + b + 0 = (a + b)(1 + 0) which is, again, a fairly trivial case. If b = 1, then we have a + 1 = (a + 1)(1). Again, this is pretty trivial.

(iii) Finally, if a ≠ 0 ≠ (1 - c - d), then d ≠ 0 and b ≠ 1. In that case we have that for any choice of b, c and d, $a = \frac{d(b-1)}{1-c-d}$. This gives a number of non-trivial solutions. For example, if b = 9, c = 2 and d = -5, then a = -10 and both sides of the equation equal 3.

4.) $(a + b)^n = a^n + b^n$

For n = 1, this is clearly true. For n > 1, it is false due to the lack of the terms $\sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} b^i$.

For n = 2, however, the "method" is valid if one is working in a ring of characteristic 2. That is obviously not going to be the case in a lower level algebra class but it does lead to an interesting question for other values of n. It turns out that the "theorem" will work if something greater than 1 divides the binomial coefficients for i = 1 to (n-1).

A little playing with Excel, for n from 2 to 53 showed a pattern. Only those n such that, for a prime, p, $n = p^k$ have a greatest common divisor greater than 1 for the necessary coefficients. Below is the theorem.

Theorem - $\gcd\left\{\binom{n}{i} \middle| \, 0 < i < n\right\} > 1$ if and only if n = p^k, k > 0, for some prime, p.

Further, $\gcd\left\{\binom{p^k}{i} \middle| \, 0 < i < p^k\right\} = p$.

So, the only cases where this "method" will be valid is in a ring of prime power characteristic. In a ring of characteristic p^k, $(a + b)^p = a^p + b^p$. Of course, the way the student uses this "method" will fail if $b = p^k$, since then $(a + b)^p = a^p + b^p = a^p$.

TRIGONOMETRY

1.) sin (nx) = n sin x and csc (nx) = n csc x

It is easy to see that this will work if n = -1, 0 or 1. If |n| > 1, then n sin x will occasionally be greater than 1 which sin (nx) can never be. If 0 < |n| < 1, then n sin x will never equal 1 while sin (nx) will. Therefore -1, 0 and 1 are the only solutions.

For cosecant, n = 0 fails due to the discontinuity of cosecant at 0.

4.) sin (x + y) = sin x + sin y

In this case, it would be necessary that

$$\sin x \cos y + \sin y \cos x = \sin x + \sin y$$
$$\sin x \cos y - \sin x = \sin y - \sin y \cos x$$
$$\sin x (\cos y - 1) = \sin y (1 - \cos x)$$
$$-\frac{1-\cos y}{\sin y} = \frac{1-\cos x}{\sin x}$$
$$-\tan\frac{y}{2} = \tan\frac{x}{2}$$
$$\tan\frac{-y}{2} = \tan\frac{x}{2}$$

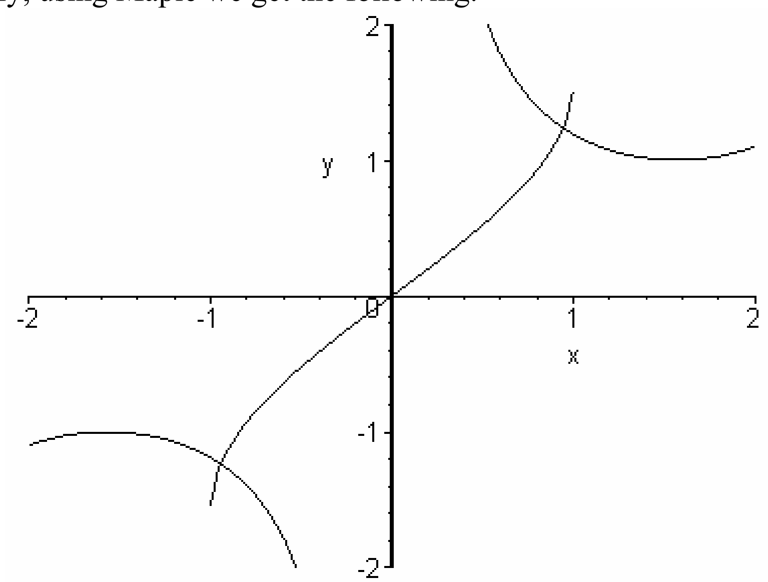
This will work as long as -y/2 and x/2 differ by an integer multiple of π. That is, for an integer, n, -y/2 = x/2 + n π or -y = x + 2nπ. To see this, consider the following.

$$\begin{aligned}\sin x \cos(-x-2n\pi) + \sin(-x-2n\pi)\cos x &= \sin x \cos(x+2n\pi) - \sin(x+2n\pi)\cos x \\ &= \sin x \cos x - \sin x \cos x \\ &= 0\end{aligned}$$

$$\sin x + \sin(-x-2n\pi) = \sin x - \sin(x+2n\pi) = \sin x - \sin x = 0$$

10.) $\sin^{-1} x = (\sin x)^{-1}$

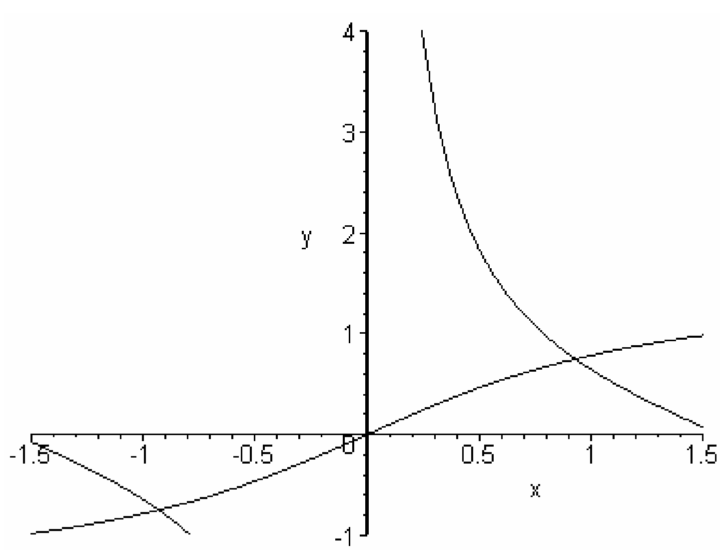
Graphically, using Maple we get the following.



Using Maple to solve we get x = ±0.9440390666.

12.) $\tan^{-1} x = (\tan x)^{-1}$

In this case, we will have infinitely many solutions since arctan x is defined for all real numbers. Restricting ourselves to x from -π/2 to π/2, graphically, using Maple we get the following.



Using Maple to solve we get x = ±0.9283948601.

LOGARITHMS

In all of the following, except number 6, assume A and B are both positive. Also, in all of these it will turn out that there are infinitely many cases where the "method" will work. However, in each case, the probability that randomly chosen numbers will work will be 0.

1.) $\log_n (AB) = (\log_n A)(\log_n B)$

In order for this to work we must have

$$\log_n A + \log_n B = (\log_n A)(\log_n B)$$

If A = n, we have $1 + \log_n B = (\log_n B)$ which is impossible so assume A is not n.

$$\log_n A = (\log_n A)(\log_n B) - \log_n B$$

$$\log_n A = \log_n B (\log_n A - 1)$$

$$\log_n B = \frac{\log_n A}{\log_n A - 1}$$

Then, for given an choice for A, this will work if $B = n^{\frac{\log_n A}{\log_n A - 1}}$.

For example, if n = 5, and A = 125, we have $B = 5^{3/2}$ giving

$$(\log_5 125)(\log_5 (5^{3/2})) = 3\ (3/2) = 9/2$$

and

$$\log_5 125 + \log_5 (5^{3/2}) = 3 + 3/2 = 9/2.$$

2.) $\log_n (A + B) = \log_n A + \log_n B$

In order for this to be true, we must have $\log_n (A + B) = \log_n (A B)$.

If A = 1, this fails so we will assume A is not 1.Since the logarithm is one to one, that means A + B = A B. That is, A = AB - B or A = B (A - 1). Thus, B = A / (A - 1). The only solution for integers A and B is A = B = 2. However, if we do not require that both be integers there are infinitely many solutions.

FORMULAS

The Distance Formula

1.) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$

In order for this to work we need the following.

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$$
$$-2\,x_1\,x_2 - 2\,y_1\,y_2 = 2\,x_1\,x_2 + 2\,y_1\,y_2$$
$$0 = 4\,x_1\,x_2 + 4\,y_1\,y_2$$
$$0 = x_1\,x_2 + y_1\,y_2$$
$$-x_1\,x_2 = y_1\,y_2$$

We end up with a lot of pairs of points for which this "method" will work. For example, the distance between (9, 3) and (2, -6) is $\sqrt{130}$ using either this "method" or the correct formula.

CALCULUS

1.) $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$

$$\left(\frac{f}{g}\right)' = \frac{f'}{g'}$$
$$\frac{g\,f' - f\,g'}{g^2} = \frac{f'}{g'}$$
$$f'\,g\,g' - f\,(g')^2 = f'\,g^2 \qquad (*)$$

Consider the following special cases.

I. Suppose f(x) = e^{ax}. Using (*) with this function gives the following.

$$ae^{ax}\,g\,g' - e^{ax}\,(g')^2 = ae^{ax}\,g^2$$
$$a\,g\,g' - (g')^2 = a\,g^2$$

The solution is:

$$g(x) = e^{\frac{ax}{2} + \frac{x}{2}\sqrt{a(a-4)} - \frac{Aa}{2} - \frac{A}{2}\sqrt{a(a-4)}} \quad or$$

$$g(x) = e^{\frac{ax}{2} - \frac{x}{2}\sqrt{a(a-4)} - \frac{Aa}{2} + \frac{A}{2}\sqrt{a(a-4)}}$$

II. Suppose g(x) = e^{ax}. Using (*) with this function gives the following.

$$ae^{2ax}\ f' - f\ a^2\ e^{2ax} = e^{2ax}\ f'$$

$$a\ f' - \ f\ a^2 = f'$$

$$f'(1-a) + a^2\ f = 0$$

$$f' \ + \frac{a^2}{1-a}\ f = 0$$

The solution would be $f(x) = A\, e^{\frac{x^3}{3(a-1)}}$.

III. Suppose f(x) = x^n. Using (*) with this function gives the following.

$$n\ x^{n-1}\ g\ g' - x^n\ (g')^2 = n\ x^{n-1}\ g^2$$

$$n\ g\ g' - x\,(g')^2 = n\ g^2$$

Using Maple to solve this gives the following solutions:

$$g(x) = 0$$

$$g(x) = A\, e^{\sqrt{n\,(n-4x)}}\ \left(\sqrt{n\,(n-4x)} - n\right)^{n/2}\ \left(n + \sqrt{n\,(n-4x)}\right)^{-n/2}\ x^{n/2}$$

$$g(x) = A\, e^{-\sqrt{n\,(n-4x)}}\ \left(\sqrt{n\,(n-4x)} - n\right)^{-n/2}\ \left(n + \sqrt{n\,(n-4x)}\right)^{n/2}\ x^{n/2}$$

IV. Suppose g(x) = x^n. Using (*) with this function gives the following.

$$f'\, x^n\, n\, x^{n-1} - f\,\left(n\, x^{n-1}\right)^2 = f'\, x^{2n}$$
$$f'\, n\, x^{2n-1} - f\, n^2\, x^{2n-2} = f'\, x^{2n}$$
$$f'\, n\, x - f\, n^2 = f'\, x^2$$
$$f'\, n\, x - f'\, x^2 = f\, n^2$$
$$f'\left(n\, x - x^2\right) = f\, n^2$$
$$\frac{f'}{f} = \frac{n^2}{nx - x^2}$$

By calculus 2 methods we get $f(x) = A\, x^n / (x - n)^n$

3.) $$\int f(x)\, g(x)\, dx = \left(\int f(x)\, dx\right)\left(\int g(x)\, dx\right)$$

Differentiating both sides gives $f(x)\, g(x) = g(x) \int f(x)\, dx + f(x) \int g(x)\, dx$

I. Suppose $f(x) = e^{ax}$.

$$e^{ax}\, g(x) = g(x)\, \frac{1}{a}\, e^{ax} + e^{ax} \int g(x)\, dx$$
$$g(x) = \frac{g(x)}{a} + \int g(x)\, dx$$
$$\frac{a-1}{a}\, g(x) = \int g(x)\, dx$$
$$g(x) = C\, e^{\frac{a}{a-1}x}$$

II. Suppose $f(x) = \cos(ax)$

$$\cos(ax)\ g(x) = g(x)\ \frac{1}{a}\ \sin(ax) + \cos(ax) \int g(x)\ dx$$

$$g(x)\left(\cos(ax) - \frac{1}{a}\ \sin(ax)\right) = \cos(ax) \int g(x)\ dx$$

$$g(x)\left(1 - \frac{1}{a}\ \tan(ax)\right) = \int g(x)\ dx \qquad \textit{differentiating}$$

$$g'(x)\left(1 - \frac{1}{a}\ \tan(ax)\right) + g(x)\left(-\sec^2(ax)\right) = g(x)$$

$$g'(x)\left(1 - \frac{1}{a}\ \tan(ax)\right) = g(x)\left(1 + \sec^2(ax)\right)$$

$$\frac{g'(x)}{g(x)} = \frac{1 + \sec^2(ax)}{1 - \frac{1}{a}\ \tan(ax)}$$

Using Maple gives the following solution.

$$g(x) = \exp\left(\frac{\ln\left(\sec^2\left(\frac{ax}{2}\right)\right)}{1+a^2} + \frac{a^2\ x}{1+a^2} - \frac{\ln\left(a\ \tan^2\left(\frac{ax}{2}\right) - a + 2\ \tan\left(\frac{ax}{2}\right)\right)}{1+a^2} - \frac{a^2\ \ln\left(a\ \tan^2\left(\frac{ax}{2}\right) - a + 2\ \tan\left(\frac{ax}{2}\right)\right)}{1+a^2} + \ln\left(\tan\left(\frac{ax}{2}\right) - 1\right) + \ln\left(\tan\left(\frac{ax}{2}\right) + 1\right)\right)$$

III. Suppose g(x) = x^n.

$$f(x)\ x^n = x^n \int f(x)\ dx + f(x)\ \frac{x^{n+1}}{n+1}$$

$$f(x) = \int f(x)\ dx + f(x)\ \frac{x}{n+1} \qquad \textit{differentiating}$$

$$f'(x) = f(x) + f'(x)\ \frac{x}{n+1} + f(x)\ \frac{1}{n+1}$$

$$(n+1)\ f'(x) = (n+1)\ f(x) + x\ f'(x) + f(x)$$

$$(n+1-x)\ f'(x) = (n+2)\ f(x)$$

$$\frac{f'(x)}{f(x)} = \frac{n+2}{n+1-x}$$

$$f(x) = A\,(-n-1+x)^{-n-2}$$

CALCULATOR

Some problems students encounter arise during work with a calculator. Frequently this will be due to not remembering parentheses.

2.) $\frac{a}{b+c} = \frac{a}{b} + c$

As it turns out, this one will produce a correct answer more frequently than its companion error in #1. Obviously b ≠ 0 ≠ (b + c).

$$\frac{a}{b+c} = \frac{a}{b} + c$$
$$a\,b = a\,(b+c) + b\,c\,(b+c)$$
$$a\,b = a\,b + a\,c + b^2\,c + b\,c^2$$
$$0 = a\,c + b^2\,c + b\,c^2 \qquad \textit{works if } c = 0 \qquad \textit{assume } c \neq 0$$
$$0 = a + b^2 + b\,c$$
$$a = -\,b^2 - b\,c$$

So, given any choice for b and c (with b ≠ 0 ≠ (b + c)), there will be an a that works. For example, if b = 3 and c = 4, we get a = -21 and both $\frac{a}{b+c}$ and $\frac{a}{b} + c$ equal -3.

ABSOLUTE VALUES AND ROOTS

Roots often cause students great problems. These "methods" will be similar to the ones for powers dealt with in another section but they are interesting enough to be handled separately.

1.) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Clearly this is true if either a or b is 0. It turns out that this is also a necessary condition.

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$
$$\left(\sqrt{a+b}\right)^2 = \left(\sqrt{a} + \sqrt{b}\right)^2$$
$$a + b = a + 2\sqrt{ab} + b$$
$$0 = 2\sqrt{ab}$$

So this turns out to be only trivially correct.

2.) $\sqrt[3]{a+b} = \sqrt[3]{a} + \sqrt[3]{b}$

Again it is clear that a = 0 or b = 0 will suffice.

$$\sqrt[3]{a+b} = \sqrt[3]{a} + \sqrt[3]{b}$$
$$\left(\sqrt[3]{a+b}\right)^3 = \left(\sqrt[3]{a} + \sqrt[3]{b}\right)^3$$
$$a + b = a + 3\sqrt[3]{a^2 b} + 3\sqrt[3]{a b^2} + b$$
$$0 = 3\sqrt[3]{a^2 b} + 3\sqrt[3]{a b^2}$$
$$-\sqrt[3]{a^2 b} = \sqrt[3]{a b^2}$$
$$-a^2 b = a b^2 \qquad \text{assuming } a \neq 0 \neq b$$
$$-a = b$$

This time we have non-trivial solutions.

3.) $\sqrt[4]{a+b} = \sqrt[4]{a} + \sqrt[4]{b}$

Again, a = 0 or b = 0 will suffice.

$$\sqrt[4]{a+b} = \sqrt[4]{a} + \sqrt[4]{b}$$
$$\left(\sqrt[4]{a+b}\right)^4 = \left(\sqrt[4]{a} + \sqrt[4]{b}\right)^4$$
$$a + b = a + 4\sqrt[4]{a^3\,b} + 6\sqrt[4]{a^2\,b^2} + 4\sqrt[4]{a\,b^3} + b$$
$$0 = 4\sqrt[4]{a^3\,b} + 6\sqrt[4]{a^2\,b^2} + 4\sqrt[4]{a\,b^3}$$
$$0 = 2\sqrt[4]{a^3\,b} + 3\sqrt[4]{a^2\,b^2} + 2\sqrt[4]{a\,b^3} \qquad \text{and after a lot of algebra}$$
$$0 = a\,b\,(16\,b^2 + 31\,a\,b + 16\,a^2)$$

We already knew about a = 0 and b = 0. Checking the other solutions shows they are extraneous roots.

Biography

Fred Worth received his B.S. in Mathematics from Evangel College in Springfield, Missouri in 1982. He received his M.S. in Applied Mathematics in 1987 and his Ph.D. in Mathematics in 1991 from the University of Missouri at Rolla. He has been teaching at Henderson State University since August, 1991.