Ballistics on Non-Level Ground

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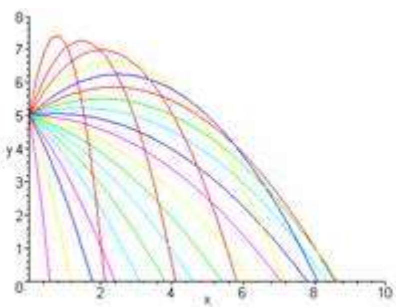
Abstract

Imagine a projectile launched from some height and hitting the ground at possibly some other elevation. Assuming there are no obstructions, no air resistance, and that the acceleration of gravity is constant, the set of possible trajectories will be family of parabolas. Here is a picture of possible trajectories where the initial height is 5 and the ground is at elevation 0. These are similar to the trajectories seen when an exploding fireworks display sends cinders in all directions.



Two main problems are addressed in this paper. The first is to find the initial angle that will maximize the range, that is, the horizontal distance. This problem was brought to my attention by an undergraduate from Florida named William Deich. His physics professor told the class that the angle for maximizing the range is not 45° when the initial height is not zero. He went home to prove his professor wrong and found the mathematics to be complicated. The second problem is to determine the distribution of the particles hitting the ground.

Maximizing the Range

Assume that the initial y coordinate is h and final y coordinate is 0. Then the position of the projectile at time t is (x, y) is given by

$$x = v(\cos\theta)t \text{ and } y = -\frac{1}{2}gt^2 + v(\sin\theta)t + h$$

where g is the acceleration of gravity and q is the initial angle.

Solve $y=0$ for t to get

$$t=\frac{v\sin\theta\pm\sqrt{v^2\sin^2\theta+2gh}}{g}$$

We want the larger of the times, so drop the negative root. Substitute this into x to get the range as a function of q and h.

$$x=\frac{v^2\sin\theta\cos\theta+v(\cos\theta)\sqrt{v^2\sin^2\theta+2gh}}{g}$$

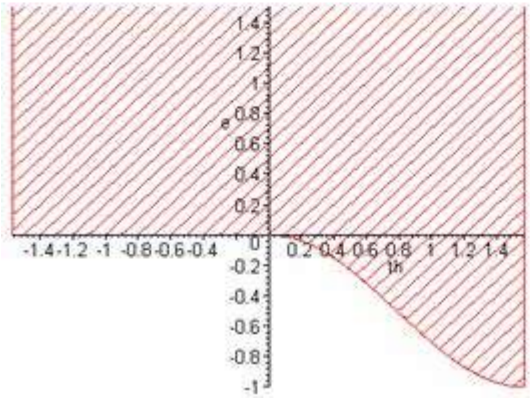
Make the substitution $e=\frac{2gh}{v^2}$ and simplify to get

$$x=\frac{v^2}{g}\left[\frac{1}{2}\sin 2\theta+\cos\theta\sqrt{\sin^2\theta+e}\right]$$

Making this substitution simplifies the algebra involved the following analysis. If $h=0$, then $e=0$, so this formula simplifies to $x=\frac{v^2}{g}\sin 2\theta$ which is maximized when $\theta=45°$. In general, you can see that if there is a q that maximizes x that it will depend only on e. Note that e is dimensionless and is the ratio of the initial potential energy to the initial kinetic energy of the projectile. (m is the mass of the projectile.)

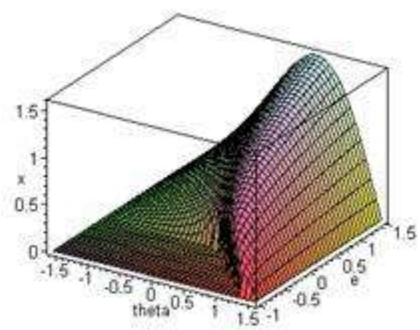
$$e=\frac{mgh}{\frac{1}{2}mv^2}$$

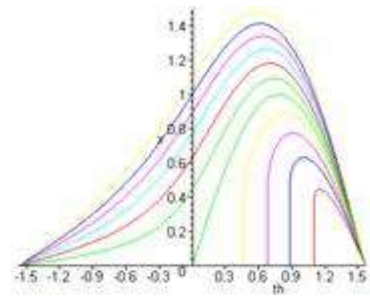
Like all good students, when we write down a function, we should be aware of its domain. So what is the domain of x as a function of q and e? Without loss of generality, we can assume that q is between -90° and 90°, and that e can be arbitrarily large. If $h<0$, then the projectile is shooting out of a hole up onto a plateau. In order to escape, the initial kinetic energy cannot be less than the initial potential energy. That is, $\frac{1}{2}mv^2\geq mg(-h)$. It follows that $e\geq -1$. This lower bound on e can be improved: If $-90°<\theta\leq 0$, then we are shooting down, so h must be non-negative, and hence $e\geq 0$. If $0\leq\theta<90°$, then the interested reader can show using Calculus that the maximum y is $y=\frac{v^2\sin^2\theta}{2g}+h$. Since this cannot be negative, it follows that $e\geq -\sin^2\theta$. The domain of x is a strip in the θe -plane as shown in the figure:



The boundary in 4th quadrant is $e = -\sin^2 \theta$ or equivalently, $\theta = \sin^{-1} \sqrt{-e}$.

The first graph below is x as a function of (θ, e). The flat part of the graph in the foreground is where x is undefined. The second graph is a graph of x versus q for values of e ranging from -.8 to 1.2.





For each $e > 0$, $\theta \to x$ is a continuous function that is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and zero at $\pm\frac{\pi}{2}$. If $e = 0$, then $\theta \to x$ is a continuous function that is positive on $(0, \frac{\pi}{2})$ and zero at the endpoints. For each $e \in (-1, 0)$, $\theta \to x$ is a continuous function that is positive on $(\sin^{-1}\sqrt{-e}, \frac{\pi}{2})$ and zero at $\frac{\pi}{2}$. At the left endpoint, $\sin^{-1}\sqrt{-e}$, x has the positive value $\sqrt{-e}\sqrt{e+1}$, but less than the max. Note that in the above plots, there is a gap at the left side of the x versus q curve when $e < 0$. Hence, for every value of $e > -1$, there is a q that maximizes x. The above plots indicate that this maximum is unique.

Now to find the critical point. Differentiate x to get

$$\frac{dx}{d\theta} = \frac{v^2}{g}\left[\cos 2\theta - \sin\theta\sqrt{\sin^2\theta + e} + \frac{\cos^2\theta \sin\theta}{\sqrt{\sin^2\theta + e}}\right]$$

Set this derivative to zero and simplify:

$$(\cos 2\theta)\sqrt{1-\cos^2\theta + e} = (\sin\theta)(1 - 2\cos^2\theta + e)$$

Square both sides to eliminate the radical, and write in terms of $\cos\theta$ to get

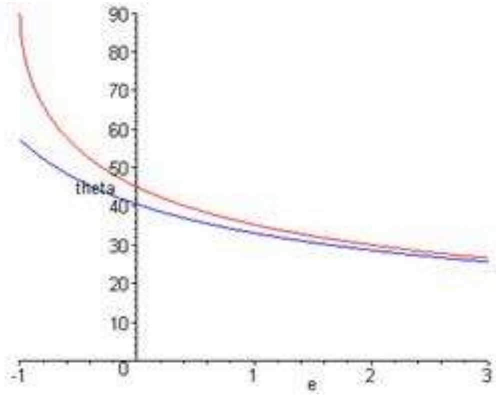
$$\left(2\cos^2\theta - 1\right)^2\left(1-\cos^2\theta + e\right) = \left(1-\cos^2\theta\right)\left(1 - 2\cos^2\theta + e\right)^2$$

Expand this out and simplify to get

$$\cos^2\theta = \frac{1+e}{2+e} \text{ or } \cos\theta = \pm\sqrt{\frac{1+e}{2+e}}$$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we discard the negative root and find the critical point is $\theta_x = \cos^{-1}\sqrt{\frac{1+e}{2+e}}$. If θ_x is in the domain of x, then being the only critical point, it must be the q which maximizes x. If $e = 0$, then $\theta_x = \cos^{-1}\frac{1}{\sqrt{2}}$, which agrees with the well-known result that 45° maximizes the range if $h = 0$. If $e \geq 0$, then the $\theta_x \in \left(0, \frac{\pi}{2}\right)$ and hence is in the domain. If $-1 < e < 0$, then we leave as an exercise for the reader to show that θ_x is in the domain of x, that is, $\sin^{-1}\sqrt{-e} < \theta_x < \frac{\pi}{2}$. The maximum value of x (the value of x at θ_x) is $\sqrt{e+1}$.

The upper red curve below is a graph of θ_x.



Notice that $e=-1$ means the projectile must be thrown pointing straight up ($\theta_x = 90°$) to escape the hole. Of course, θ_x crosses the vertical axis at $45°$. As $e \to \infty$, θ_x approaches zero. In fact, θ_x is asymptotic to $\theta = \frac{1}{\sqrt{2+e}}$ as $e \to \infty$. This second function is the lower function drawn in blue in the above graph. Since this function converges to zero slowly, William was almost right when he challenged his physics professor: The angle which maximizes the range will usually be a little less than $45°$.

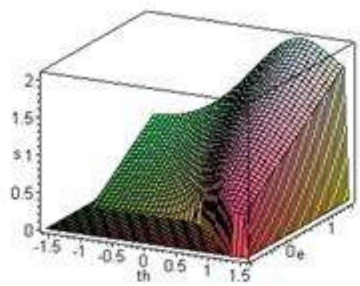
To prove that $\theta_x \approx \frac{1}{\sqrt{2+e}}$, use L'Hôpital's rule to show $\lim_{e\to\infty} \frac{\theta_x}{1/\sqrt{2+e}} = 1$. Then use L'Hôpital's rule again to show that $\lim_{e\to\infty}\left[e^{3/2}\left(\theta_x - \frac{1}{\sqrt{2+e}}\right)\right] = \frac{1}{6}$. The first computation is simplified if the substitution $u = \frac{1}{2+e}$ is made.

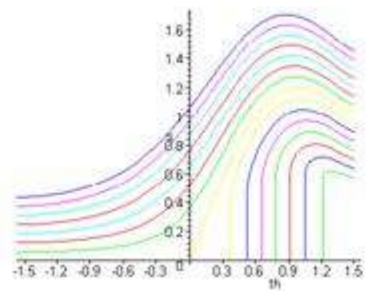
Maximizing the Arc Length

Besides maximizing the range, what other problems can be posed? Clearly, the time aloft is maximized when $\theta = \frac{\pi}{2}$. We will now attempt to maximize the arc length of the trajectory. The time aloft we found earlier written in terms of e is $t_a = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + e}\right)$. The arc length of the projectile path $s = \int_0^{t_a} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ simplifies to

$$s = \int_0^{t_a} \sqrt{(v\cos\theta)^2 + (-gt + v\sin\theta)^2}\ dt = \int_0^{t_a} \sqrt{gt^2 - (2gv\sin\theta)t + v^2}\ dt = v\int_0^{t_a} \sqrt{\left(\frac{gt}{v}\right)^2 - (2\sin\theta)\left(\frac{gt}{v}\right) +}$$

Make the change of variable $u = \frac{gt}{v}$ to get $s = \frac{v^2}{g}\int_0^{\sin\theta+\sqrt{\sin^2\theta+e}} \sqrt{u^2 - (2\sin\theta)u + 1}\, du$. As in the solution for maximizing the range, this shows that if there is a q that maximizes s, that it only depends only on e. To simplify the notation, we'll drop the $\frac{v^2}{g}$ in front. Here is a graph of s versus q and e, and s versus q for e ranging from –1 to 1.5.





Note that these graphs are similar to the graphs of the range. The steep parts of the curves in the lower right corner of the second graph are really gaps. That is, when $e < 0$, the minimal angle q which will allow the projectile to escape the hole, $\theta = \sin^{-1}\sqrt{-e}$, gives a positive arc length.

A hard Calculus II integration yields

$$s = \frac{1}{2}\left[ab + \sin\theta + (\cos^2\theta)\ln\left(\frac{a+b}{1-\sin\theta}\right)\right]$$

where $a = \sqrt{\sin^2\theta + e}$ and $b = \sqrt{e+1}$. The derivative of s with respect to q is

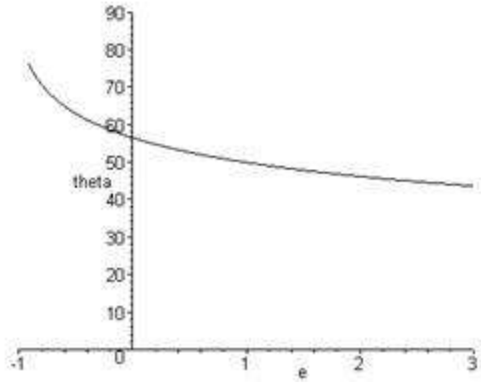
$$\frac{ds}{d\theta} = \frac{1}{2}\left[\frac{b\sin\theta\cos\theta}{a} + \cos\theta - \sin(2\theta)\ln\left(\frac{a+b}{1-\sin\theta}\right) + (\cos^3\theta)\left(\frac{\sin\theta}{a(a+b)} + \frac{1}{1-\sin\theta}\right)\right]$$

If $e = 0$, then s and $ds/d\theta$ simplify to

$$s = \sin\theta + (\cos^2\theta)\ln(\sec\theta + \tan\theta) \text{ and } \frac{ds}{d\theta} = -2(\cos\theta)[(\sin\theta)\ln(\sec\theta+\tan\theta) - 1],$$

respectively. Setting the derivative to zero gives $(\sin\theta)\ln(\sec\theta + \tan\theta) = 1$. It does not seem to be possible to solve $ds/d\theta = 0$ analytically even for the special case when $e = 0$, so we will have to use numerical methods.

Let θ_s be the q which maximizes s. The graph of θ_s versus e is



The θ_s values used to create this graph were obtained by solving $ds/d\theta = 0$ numerically using Maple. Here is a table of a few values of θ_s and e.

e	-.999	-.95	0	100	10^4	10^6
θ_s	≈ 88°	79.0°	56.5°	25.5°	12.7°	8.5°

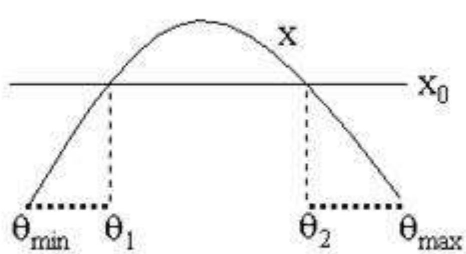
As $e \to -1$, θ_s appears to approach 90°. θ_s appears to approach 0 very slowly as $e \to \infty$.

Distribution of Range

Assuming the possible values of q are uniformly distributed, what is the distribution of the range of x? We will first find the cumulative distribution function for x.

Range when shooting from a hill

Let Fpos be the cumulative distribution function for when $e > 0$. The possible angles q range from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, an interval of length p. Let $\theta_1 < \theta_2$ be the two values of q where $x = x_0$ if $x_0 < \text{maximum x value} = \sqrt{e+1}$.



Then

$$Fpos(x_0) = P(x \le x_0) = \frac{(\theta_1 - \theta_{min}) + (\theta_{max} - \theta_2)}{\theta \text{ range}} = \frac{1}{\pi}\left[\left(\theta_1 - -\frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \theta_2\right)\right] = \frac{1}{\pi}[\theta_1 - \theta_2 + \pi]$$

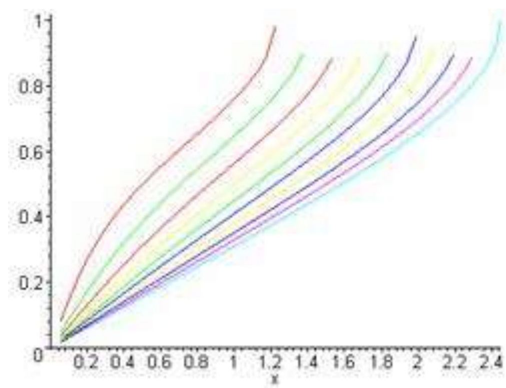
For simplicity, assume that $v^2 / g = 1$ so that the values of θ_1 and θ_2 can be found by solving

$$\frac{1}{2}\sin 2\theta + \cos\theta\sqrt{\sin^2\theta + e} = x_0$$

for q. Isolate the radical, square both sides, and simplify to get $x_0^2 \tan^2\theta - 2x_0 \tan\theta + (x_0^2 - e) = 0$. Use the quadratic formula to get

$$\theta = \tan^{-1}\left[\frac{1}{x_0}\left(1 \pm \sqrt{e + 1 - x_0^2}\right)\right]$$

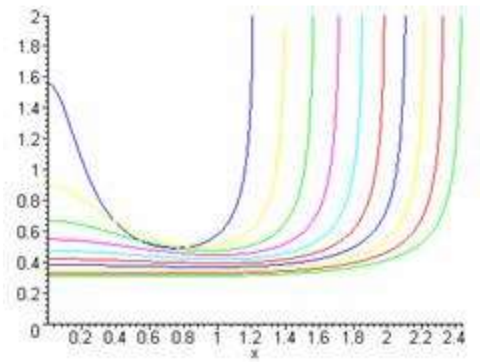
θ_1 and θ_2 are the minus and plus roots, respectively. The graph of Fpos for values of e ranging from 0.5 to 5 is given here:



Note that the cumulative distribution functions are zero if $x < 0$ and equal to 1 if $x > \sqrt{e+1}$. Differentiate Fpos and simplify to obtain the probability density function

$$fpos(x) = \frac{2}{\pi}\frac{2x^2 + e + e^2}{(4x^2 + e^2)\sqrt{1 - x^2 + e}}$$

The graph of this function for values of e ranging from 0.5 to 5 is



Note that the cumulative distribution has a vertical tangent at $x = \sqrt{e+1}$. If e is large, then it is more likely for the projectile to fall near the maximum. If $e > 0$ is small, then fpos is also large near $x = 0$. If a fireworks display is exploding high enough, it would be safer to stand under it than near the maximum range of the sparks! (That is, assuming that all the projectiles have the same velocity.) If the explosion is too low, the U-shaped distribution shows that it would not be safe to stand directly under the explosion…

Range when shooting from level ground

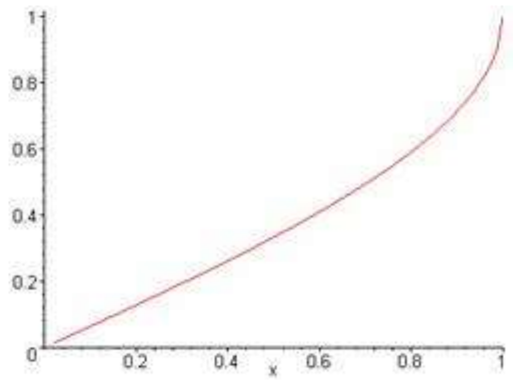
Let F0 be the cumulative distribution function when $e = 0$. The possible angles q range from 0 to $\frac{\pi}{2}$. Let θ_1 and θ_2 be the same angles as before. Then

$$F0(x_0) = \frac{2}{\pi}\left[\theta_1 + \frac{\pi}{2} - \theta_2\right]$$

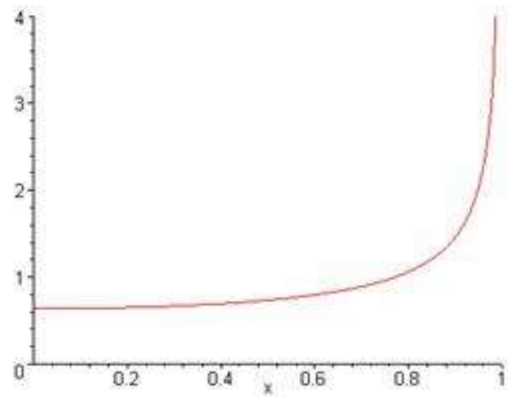
Differentiate and simplify to get

$$f0(x) = \frac{2}{\pi\sqrt{1-x^2}}$$

Here are the graphs of the cumulative and density functions we just derived:



Cumulative Distribution Function F0



Probability Density Function f0

An application for this case is that fireworks shooting randomly from the ground are more likely to fall near the maximum range.

Range when shooting from a hole

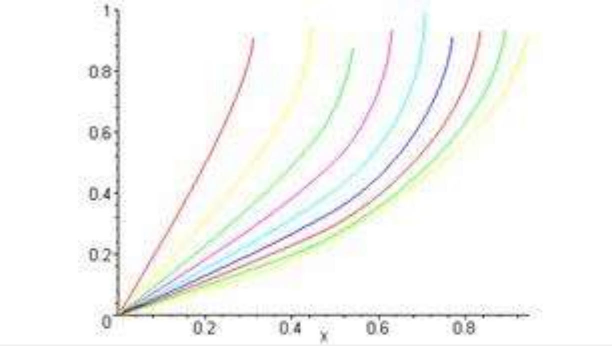
Let Fneg be the cumulative distribution function when $e < 0$. The possible angles q range from $\sin\sqrt{-e}$ to $\frac{\pi}{2}$. Let θ_1 and θ_2 be the same angles as before. There are two possible cases when computing Fneg because x is not zero at the smallest possible angle, $\sin\sqrt{-e}$.

$$Fneg(x_0) = \begin{cases} \dfrac{\theta_1 + \pi/2 - \theta_2}{\pi/2 - \sin^{-1}\sqrt{-e}} & \text{if } \sqrt{-e}\sqrt{1+e} < x < 1 \\ \dfrac{\pi/2 - \theta_2}{\pi/2 - \sin^{-1}\sqrt{-e}} & \text{if } 0 < x \le \sqrt{-e}\sqrt{1+e} \end{cases}$$

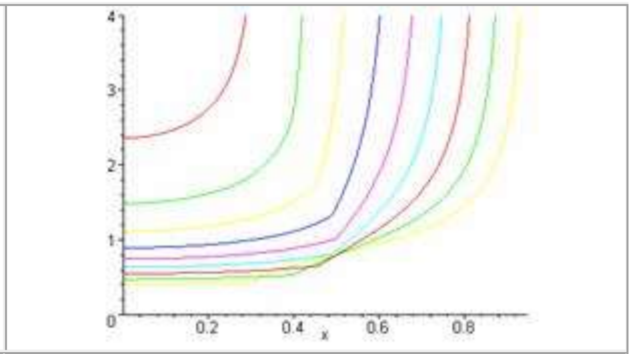
The derivative of the first piece is the almost the same as the derivative of Fpos. The derivative of Fneg simplifies to

$$fneg(x) = \begin{cases} \dfrac{2}{\pi/2 - \sin^{-1}\sqrt{-e}} \dfrac{2x^2 + e + e^2}{\left(4x^2 + e^2\right)\sqrt{1 - x^2 + e}} & \text{if } \sqrt{-e}\sqrt{1+e} < x < 1 \\ \dfrac{1}{\pi/2 - \sin^{-1}\sqrt{-e}} \dfrac{\sqrt{1 - x^2 + e} + 1 + e}{\sqrt{1 - x^2 + e}\left[2 + 2\sqrt{1 - x^2 + e} + e\right]} & \text{if } 0 < x < \sqrt{-e}\sqrt{1+e} \end{cases}$$

Here are the graphs of the cumulative and density functions we just derived for values of e ranging from -.9 to -.1:



Cumulative Distribution Function Fneg



Probability Density Function fneg

As before, the probability density blows up near the maximum x value. One application is if an enemy is firing a cannon randomly at you while you are on a plateau. It is safer to wait near the edge of the cliff closer to the enemy, than near the cannon’s maximum range.

Biographical Sketches

Michael Lloyd received his B.S. in Chemical Engineering in 1984 and his Ph.D. in Mathematics from Kansas State University in Manhattan. He has presented papers at meetings of the Mathematical Association of America, the Arkansas Conference on Teaching, and the American Mathematical Society. He has been at Henderson State University since August 1993.

William Deich was a student at Broward Community College in Davie, FL when he suggested this problem to Michael Lloyd. He has since recieved his A.A. degree in Physics with departmental awards in mathematics and physical science from BCC in 2000 and moved on the the University of Florida in Gainesville, FL where he recieved his B.S. in computer science magna cum laudi in 2005.