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Evaluation of Indefinite Integrals of $sec\ x$ and $csc\ x$ by the MUTOBO Method

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Abstract

In most Calculus books, one finds well-known tricks for evaluating $\int \sec x\, dx$ and $\int \csc x\, dx$. In this article, we give another method of evaluating these integrals.

Introduction

In most good Calculus books, one finds well-known tricks for evaluating $\int \sec x\, dx$ and $\int \csc x\, dx$, namely

$$\int \sec x\, dx = \int \sec x \cdot 1\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
$$= \int \frac{1}{u} du, \text{where } u = \sec x + \tan x$$
$$= \ln|u| + C = \ln|\sec x + \tan x| + C,$$

and

$$\int \csc x\, dx = \int \csc x \cdot 1 dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} dx$$
$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx = \int \frac{1}{u} du, \text{where } u = \cot x + \csc x$$
$$= -\ln|u| + C = -\ln|\csc x + \cot x| + C.$$

Some Calculus II instructors, the author included, have had a difficult time giving a convincing answer when students ask them the following questions: How did you come up with this trick? Is there another method to evaluate these integrals?

In this article, we give one additional method of evaluating these integrals. We call this method the MUTOBO Method. Before we embark on this task, we need some preliminaries.

Preliminaries

In this section, we state and prove some key results that we will use in the MUTOBO Method to evaluate $\int \sec x\, dx$ and $\int \csc x\, dx$. We begin with

Lemma 1 *If x is a real number, then*

$$\frac{1+\sin x}{1-\sin x}=(\sec x+\tan x)^2. \quad (1)$$

Proof.

$$\frac{1+\sin x}{1-\sin x}=\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}=\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}=\frac{1+2\sin x+\sin^2 x}{1-\sin^2 x}$$

$$=\frac{1+2\sin x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}+\frac{2\sin x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}$$

$$=sec^2 x+2\tan x\sec x+\tan^2 x=(\sec x+\tan x)^2.$$

This completes the proof of (1).

Lemma 2 *If x is a real number, then*

$$\frac{1+\cos x}{1-\cos x}=(\csc x+\cot x)^2. \quad (2)$$

Proof.

$$\frac{1+\cos x}{1-\cos x}=\frac{1+\cos x}{1-\cos x}.\frac{1+\cos x}{1+\cos x}=\frac{(1+\cos x)^2}{(1-\cos x)(1+\cos x)}$$

$$=\frac{1+2\cos x+\cos^2 x}{1-\cos^2 x}=\frac{1+2\cos x+\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x}+\frac{2\cos x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}$$

$$=\csc^2 x+2\csc x\cot x+\cot^2 x=(\csc x+\cot x)^2.$$

This completes the proof of (2).

We are now ready to use the MUTOBO Method to evaluate the indefinite integrals $\int \sec x\, dx$ and $\int \csc x$ dx.

Evaluating Indefinite Integrals of $sec\ x$ and $csc\ x$ by the MUTOBO Method

In this section, we use the multiply-top-and-bottom (MUTOBO) method to evaluate $\int sec\ x\ dx$ and $\int \csc x\, dx$. We begin with evaluating $\int \sec x\, dx$.

$$\int \sec x dx=\int \frac{1}{\cos x}dx=\int \frac{1}{\cos x}.\frac{\cos x}{\cos x}dx=\int \frac{\cos x}{\cos^2 x}dx=\int \frac{\cos x}{1-\sin^2 x}dx$$

$$=\int \frac{1}{1-u^2}du, \text{where } u=\sin x$$

$$=\int \frac{1}{(1+u)(1-u)}du=\int \left[\frac{1/2}{1+u}+\frac{1/2}{1-u}\right]du, \text{using partial fractions}$$

$$=\frac{1}{2}\left[\int \frac{1}{1+u}du+\int \frac{1}{1-u}du\right]=\frac{1}{2}[\ln|1+u|-\ln|1-u|]+C$$

$$=\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \frac{1}{2}\ln(\sec x + \tan x)^2 + C, \text{ using Lemma 1}$$

$$= \ln\sqrt{(\sec x + \tan x)^2} + C = \ln|\sec x + \tan x| + C.$$

Next, we evaluate $\int \csc \mathrm{x}\, dx$ using the MUTOBO Method.

$$\int \csc x \,\mathrm{dx} = \int \frac{1}{\sin x} dx = \int \frac{1}{\sin x} . \frac{\sin x}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x}{1-\cos^2 x} dx$$

$$= -\int \frac{1}{1-u^2} du, \text{where } u = \cos x$$

$$= -\int \frac{1}{(1+u)(1-u)} du = -\int \left[\frac{1/2}{1+u} + \frac{1/2}{1-u}\right] du, \text{using partial fractions}$$

$$= -\frac{1}{2}\left[\int \frac{1}{1+u} du + \int \frac{1}{1-u} du\right] = -\frac{1}{2}[\ln|1+u| - \ln|1-u|] + C$$

$$= -\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C$$

$$= -\frac{1}{2}\ln(\csc x + \cot x)^2 + C, \text{ using Lemma 2}$$

$$= -\ln\sqrt{(\csc x + \cot x)^2} + C = -\ln|\csc x + \cot x| + C.$$

Concluding Remarks

To evaluate the indefinite integrals $\int \sec x \, dx$ and $\int \csc x \, dx$, one may also use the Weierstrass substitution

$$u = \tan\left(\frac{x}{2}\right), \text{ for } -\pi < x < \pi.$$

For details on the Weierstrass substitution, see Bradley and Smith [1], p. 495. We do confess that both the MUTOBO Method and the Weierstrass Substitution Method are more complicated than the tricks used in the introduction to evaluate $\int \sec x \, dx$ and $\int \csc x \, dx$. These two methods would, without a doubt, be good projects for inquisitive Calculus II students. It would be nice if every Calculus II instructor were aware of these methods!

References

[1] Bradley, G.L. & K.J. Smith, *Calculus*, Prentice-Hall, Inc., 1995.
[2] Sullivan, M., *Trigonometry: A Unit Circle Approach*, 8th Edition, Pearson, 2008.

Biographical Sketch

Lloyd Moyo received his B.Ed. (Science) in 1992 from the University of Malawi in Africa. He received his M.Sc. in Mathematics from the University of Sussex, U.K. in 1996 and his Ph.D. in Mathematics from New Mexico State University in 2006. He joined Henderson State University in fall 2012. He is a member of the American Mathematical Society (AMS), the Mathematical Association of America (MAA), Arkansas Academy of Science (AAS), and International Mathematical Union (IMU).